# Serie 6

Consider $X$ a compact Hausdorff space, $\mathcal{B}$ its $\sigma$–algebra of Borel sets, $\mathcal{H}$ a Hilbert space and a resolution of the identity $E\colon \mathcal{B} \to \mathcal{L}(\mathcal{H})$. For $f\colon X \to \mathbb{C}$, set $\|f\| := \sup_{x\in X} |f(x)|$ and define

$$\mathcal{B}^\infty(X) := \{\, f\colon X \to \mathbb{C} \mid f \text{ is Borel measurable and bounded}\,\}.$$

**Exercise 1.** Show that provided with $\|.\|$, the space $\mathcal{B}^\infty(X)$ is a $C^*$–algebra.

As defined in the lecture, for any $f \in \mathcal{B}^\infty(X)$ denote $\|f\|_\infty$ its essential supremum and define

$$N := \{\, f \in \mathcal{B}^\infty(X) \mid \|f\|_\infty = 0\,\}.$$

**Exercise 2.** For $f, g \in \mathcal{B}^\infty(X)$, show that:

1. $\|f\|_\infty \le \|f\|$,
2. $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$,
3. $\|f \cdot g\|_\infty \le \|f\|_\infty \cdot \|g\|_\infty$.

**Exercise 3.** For every $f \in \mathcal{B}^\infty(X)$ and every $g \in N$, show that $\|f+g\|_\infty = \|f\|_\infty$.

**Exercise 4.** Show that the quotient norm on $\mathcal{B}^\infty(X)/N$ is given by $\|[f]\| = \|f\|_\infty$.

**Exercise 5.** Show that $L^\infty := \mathcal{B}^\infty(X)/N$, endowed with the quotient norm, is a $C^*$–algebra and for every $[f] \in L^\infty(E)$, the spectrum of $[f]$ coincides with the essential image $EssIm(f)$ of any representation of $[f]$.