Serie 7

Exercise 1. In the context of Theorem 4.10 (Lecture 9), show that for $f \in C(Sp_A(x))$ we have

$$Sp_A(f(x)) = f(Sp_A(x)).$$

Exercise 2. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator and E the associated resolution of identity associated to it (see Corollary 5.20). For $f \in C(Sp(T))$ define, using $f : Sp(T) \to Sp(f(T))$, the map $f_*(E)(w) := E(f^{-1}(w))$ for every $w \subset Sp(f(T))$ a Borel set. Show that $f_*(E)$ is the resolution of the identity on $Sp(f(T))$ associated to $f(T)$.

Exercise 3. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator and E its associated resolution of the identity. If $0 \in Sp(T)$, then $\mathrm{Ker}(T) = \mathrm{Im}E(\{0\})$.

Exercise 4. With the notations of Exercise 3 and 2, show that $\mathrm{Ker}(f(T)) = \mathrm{Im}(E(f^{-1}(0)))$ assuming $0 \in Sp(f(T))$.

Exercise 5. In the same context, show that if $\lambda_0 \in Sp(T)$, then $\mathrm{Ker}(T - \lambda_0 Id) = \mathrm{Im}E\{\lambda_0\}$. Conclude that λ_0 is an eigenvalue of T if and only if $E\{\lambda_0\} \neq 0$. Show that if $\lambda_0 \in Sp(T)$ is an isolated point of $Sp(T)$ then λ_0 is an eigenvalue.

Exercise 6. Let $T \in \mathcal{L}(\mathcal{H})$ be a normal operator. Show that $\|T\| = \sup\{\, |\langle Tx\,,\, x\rangle| \mid \|x\| \leq 1\,\}$.

Hint. Use that $\|T\| = \|T\|_{Sp}$ so that there exists $\lambda_0 \in Sp(T)$ with $\|T\| = |\lambda_0|$. Then use that for any open set ω containing λ_0, we have $E(\omega) \neq 0$.

Exercise 7. Let $V \in \mathcal{L}(\mathcal{H})$ be a unitary operator and P the orthogonal projection of $\mathcal{H}$ onto the eigenspace of V for the eigenvalue 1. Use the Spectral Theorem to show that for every $x \in \mathcal{H}$ it holds

$$\lim_{n\to\infty} \frac{1}{n}\left(x + Vx + \cdots + V^{n-1}x\right) = P(x).$$

Hint. You can try to apply Theorem 5.16 with the sequence of functions $f_n : \lambda \in Sp(V) \to n^{-1}\sum_{k=0}^{n-1}\lambda^k$ and apply the Dominated Convergence Theorem.

This is the abstract form of Von Neumann's ergodic theorem whose origine is the following: let (X,μ) be a probability measure space, that is $\mu(X) = 1$, and let $\psi : X \to X$ be a measure preserving bijection. Consider walking along the orbit $x, \psi(x), \ldots, \psi^{n-1}(x)$ of a point x. Then the time average of a function $f : X \to \mathbb{C}$ along this orbit is

$$\lim_{n\to\infty} \frac{1}{n}\left(f(x) + f\circ\psi(x) + \cdots + f\circ\psi^{n-1}(x)\right),$$

which may or may not exists. The space average of $f \in L^1(X,\mu)$ is $\int_X f d\mu$.

Exercise 8. Show that for every $f \in L^2(X,\mu)$ the sum $\lim_{n\to\infty} n^{-1}\left(f + f\circ\psi + \cdots + f\circ\psi^{n-1}\right)$ converges in the L^2 sense to a function $g \in L^2(X,\mu)$ verifying $g\circ\psi = g$. Moreover, show that if the only ψ–invariant L^2–functions are the constants, then the above limit equals the constant function $\int_X f d\mu$ and conversely.

A transformation with the property that the only ψ–invariant L^2–functions are the constant is called ergodic.

Exercise 9. Show that the transformation $\psi\colon X \to X$ is ergodic if and only if for every $f, g \in L^2(X, \mu)$ it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle\, f \circ \psi^k \,,\, g\,\rangle = \langle\, f \,,\, \mathbb{1}\,\rangle \cdot \langle\, \mathbb{1} \,,\, g\,\rangle,$$

where $\mathbb{1}$ is the constant function at 1.

Exercise 10. Let $\mathcal{H}$ be a separable Hilbert space and $A \subset \mathcal{L}(\mathcal{H})$ an abelian C^*–subalgebra. Assume that there exists $v \in \mathcal{H}$ with $\overline{Av} = \mathcal{H}$. Let E be the resolution of identity associated to A by the Spectral Theorem. Show the following:

1. For any Borel subset $\omega \in \hat{A}$ we have $E(\omega) = 0$ if and only if $E_{v,v}(\omega) = 0$. Conclude $L^\infty(E) = L^\infty(\hat{A}, E_{v,v})$.
2. Use the isomorphism $\Delta\colon \mathcal{H} \to L^2(\hat{A}, E_{v,v})$ to show that the subalgebra
$$Z_{\mathcal{L}(\mathcal{H})}(A) := \{\, T \in \mathcal{L}(\mathcal{H}) \mid Ta = aT \text{ for every } a \in A\,\}$$
is isomorphic to $L^\infty(\hat{A}, E_{v,v})$
Hint: show that the bounded operator on $L^2(\hat{A}, E_{v,v})$ that commute with multiplication with $C(\hat{A})$ are given by multiplication with the functions from $L^\infty(\hat{A}, E_{v,v})$.
3. Conclude that he image of the C^*–subalgebra homomorphism $\psi\colon L^\infty(E) \to \mathcal{L}(\mathcal{H})$ from the Spectral Theorem is $Z_{\mathcal{L}(\mathcal{H})}(A)$ and hence is abelian.
4. Give an example of $A \subset \mathcal{L}(\mathcal{H})$ for which $Z_{\mathcal{L}(\mathcal{H})}(A)$ is not abelian.