Serie 9

Exercise 1. If A is a Banach algebra, the group $G(A)$ of invertible elements with the topology induced from A is a topological group. Check that $G(A)$ is locally compact if and only if A is finite dimensional.

Exercise 2. $\prod G_\alpha$ is a locally compact Hausdorff group if and only if all G_α's are locally compact Hausdorff and the G_α are compact except finitely many.

Exercise 3. Let (X, d) be a metric space such that all closed balls of finite radius are compact, then the group $Is(X)$ of isometries of (X, d) with compact open topology is locally compact.

Hint. You can use Arzelà–Ascoli Theorem: Let X be a topological space, Y a Hausdorff uniform space and $H \subset C(X, Y)$ an equicontinuous set of continuous functions such that $H(x)$ is relatively compact in Y for each $x \in X$. Then H is relatively compact in $H \subset C(X, Y)$.

Exercise 4. Suppose $p \equiv 1 \ mod(4)$, then construct inductively a sequence $x_n \in \mathbb{Z}/p^n\mathbb{Z}$ with

- $x_n^2 + 1 = 0$ in $\mathbb{Z}/p^n\mathbb{Z}$
- $\phi(x_n) = x_{n-1}$.

Thus $x^2 + 1 = 0$ has a solution in $\mathbb{Z}_p$.

Hint. Use that if $p \equiv 1 \ mod(4)$, then -1 is a square in $\mathbb{Z}/p\mathbb{Z}$

Exercise 5. Check that the product topology on $\mathbb{Z}_p$ as a subspace of $\prod A_n$ coincides with the topology on $\mathbb{Z}_p$ as a subspace of the metric space $\mathbb{Q}_p$.

Exercise 6. Compute $\pi_0(G)$ in each of Examples 6.3 except examples $3, 6$ and 7.

Exercise 7. Consider the three-dimensional Heisenberg group $H = R \rtimes_\eta \mathbb{R}^2$, where $eta : \mathbb{R} \to Aut(\mathbb{R}^2)$ is defined by

$$\eta(x)(y, z) = (y, z + xy),$$

for every $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Verify that the Lebesgue measure is the Haar measure of the Heisenberg group.