

5. C. The Spectral Theorem.

The spectral theorem is that for every (bounded) normal operator $T \in \mathcal{L}(\mathcal{H})$ induces in a canonical way a resolution E of the identity defined on the Borel subsets of its spectrum ~~$\sigma(T)$~~ $\sigma(T) \subset \mathbb{C}$ and that T can be reconstructed from E by an integration process as discussed in Thm 5.16.

In fact this will be a special case of a spectral theorem for abelian sub-
- C^* -algebras of $\mathcal{L}(X)$.

Theorem 5.18 Let $A \subset L(\mathbb{K})$ be an abelian sub- C^* -algebra containing the identity and let \hat{A} be its Gelfand spectrum. Then the following hold:

(1) There is a unique resolution E of the identity defined on the Borel sets of \hat{A} which satisfy for every $T \in A$

$$T = \int_{\hat{A}} \hat{T} dE \quad (\text{see Thm 5.16} \\ \text{for notation})$$

where $\hat{T} \in C(\hat{A})$ is the Gelfand transform of T .

(2) The inverse $C(\hat{A}) \rightarrow A$ of the Gelfand transform extends to a C^* -algebra isomorphism $\hat{\mathcal{I}}$ of $L^\infty(E)$

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onto a C^* -subalgebra $B \subset L(\mathcal{H})$

$B \supset A$ given by :

$$\underline{\mathbb{E}}(f) = \int_A f dE, \quad f \in L^\infty(E)$$

Explicitly : $\underline{\mathbb{E}}$ is linear, multiplicative
and satisfies $\underline{\mathbb{E}}(\bar{f}) = \underline{\mathbb{E}}(f)^*$, $\|\underline{\mathbb{E}}(f)\| = \|f\|_\infty$.

(3) B is the closure in $L(\mathcal{H})$ of the
space of $\leq n$ finite linear combinations
of the projections $E(\omega)$.

(4) If $\omega \subset \hat{A}$ is open, non-empty
then $E(\omega) \neq 0$.

(5) An operator $S \in L(\mathcal{H})$ commutes
with every $T \in A$ iff S commutes with
every projection $E(\omega)$.

Comments

(1) Let $f \in C(\hat{A})$ and assume $\|f\|_\infty = 0$. Then $E(|f|^2([0, \infty[)) = 0$ and $|f|^2([0, \infty[)$ being open (4) implies that it is empty, hence $f = 0$ and $C(\hat{A})$ injects into $L^\infty(E)$.

(2) Since A is abelian it follows from (5) that every $T \in A$ commutes with every projection $E(w)$.

(3) Let $T \in A$ and $x_0 \in \hat{A}$ and consider the open subset for $\varepsilon > 0$:

$$\omega = \{x \in \hat{A} : |\hat{T}(x) - \hat{T}(x_0)| < \varepsilon\}.$$

Then $\omega \neq \emptyset$ and by (4), $E(\omega) \neq 0$.

We have $\forall \omega \in \text{Im}(E(\omega))$:

$$\|T\omega - \hat{T}(x_0) \cdot \omega\| \leq \varepsilon \| \omega \|$$

so $\text{Im}(E(\omega))$ consists of "almost eigenvectors" of eigenvalue $\hat{T}(x_0)$.

Indeed we have

$$(T - \hat{T}(x_0) \cdot \text{Id}) E(\omega) =$$

$$= \bar{\Phi}((\hat{T} - \hat{T}(x_0) \cdot \mathbb{1}) \cdot x_\omega)$$

and hence,

$$\|(T - \hat{T}(x_0) \cdot \text{Id}) E(\omega)\| = \|(\hat{T} - \hat{T}(x_0) \cdot \mathbb{1}) x_\omega\|$$

$$\leq \varepsilon.$$

We need the following

Lemma 5.19. Let $f: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ be

sequilinear bounded in the sense that

$$M := \sup \left\{ |f(x, y)| : \|x\| = \|y\| = 1 \right\} < +\infty.$$

Then $\exists T \in \mathcal{L}(\mathcal{H})$ unique such that

$$f(x, y) = \langle Tx, y \rangle \quad \forall x, y \in \mathcal{H}.$$

Moreover $\|T\| = M$.

Proof: By scaling and sequilinearity

we get $|f(x, y)| \leq M\|x\|\|y\| \quad \forall x, y \in \mathcal{H}$.

Next, for every $y \in \mathcal{H}$, $x \mapsto f(x, y)$ is a continuous linear functional and hence

$\exists! s_y \in \mathcal{H}$ with $\langle x, s_y \rangle = f(x, y)$.

~~Also $s_y = T^*y$ for all $y \in \mathcal{H}$.~~

~~Linearity of T^* follows from linearity~~

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From the sublinearity of f we deduce
that $y \mapsto \mathcal{S}_y$ is linear. Moreover

$$\|\mathcal{S}\| = \sup_{\|y\|=1} \|\mathcal{S}_y\| = \sup_{\|y\|=1} \sup_{\|x\|=1} \langle x, \mathcal{S}_y \rangle$$

$$= \sup_{\|y\|\leq 1, \|x\|=1} |f(x,y)| = M.$$

Now define $T := \mathcal{S}^*$. □

~~$f(x) \mapsto f(x,y)$~~ and

$$\begin{aligned} \|T\| &= \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \sup_{\|y\|=1} \{|f(x,y)| \\ &= \sup_{\|x\|=1} \sup_{\|y\|=1} |f(x,y)| = M. \end{aligned}$$

□

Proof of Thm 5.18 :

Let us denote $g : C(\hat{A}) \rightarrow A$

the inverse of the Gelfand isomorphism;

\tilde{g} is then also a C^* -algebra isomorphism.

Let $x \in \mathcal{H}$ and consider :

$$C(\hat{A}) \longrightarrow \mathbb{C}$$

$$f \mapsto \langle g(f)x, x \rangle$$

It is a linear functional. Moreover

assume $f \geq 0$ and set $g := \sqrt{f}$.

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Then $f = \bar{g} \cdot g$, hence $\mathcal{J}(f) = \mathcal{J}(g) \mathcal{J}(g)$

from which follows:

$$\begin{aligned}\langle \mathcal{J}(f)x, x \rangle &= \langle \mathcal{J}(g)^* g(g)x, x \rangle \\ &= \|g(g)x\|^2 \geq 0.\end{aligned}$$

Hence there is a positive regular Borel measure $E_{x,x}$ on \hat{A} such

that:

$$\langle Tx, x \rangle = \int_{\hat{A}} T dE_{x,x}, \quad \forall T \in A, \quad \forall x \in \mathbb{R}.$$

Define now the complex measure

$$2E_{x,y} := E_{x+y, x+y} + i E_{x+iy, x+iy} - (1+i) E_{x, x} - (1+i) E_{y, y} \quad (*)$$

so that

$$\langle Tx, y \rangle = \int_{\hat{A}} T dE_{x,y} \quad \forall T \in A \quad (**)$$

$x, y \in \mathbb{R}.$

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From (**) we deduce that for every

$$f \in C(\hat{A}), (x,y) \mapsto \int_{\hat{A}} f dE_{x,y}$$

is sesquilinear which implies that
the same holds for $f \in \mathcal{B}^\infty(\hat{A})$.

Next we observe that for any $\alpha \in \mathbb{C}$
and $f \in \mathcal{B}^\infty(\hat{A})$,

$$\left| \int_{\hat{A}} f dE_{v,w} \right| \leq \|f\| \cdot E_{v,w}(\hat{A}) \\ = \|f\| \|w\|.$$

Together with (*) this implies that
the sesquilinear form $(x,y) \mapsto \int_{\hat{A}} f dE_{x,y}$
(for $f \in \mathcal{B}^\infty(\hat{A})$) is bounded and
hence there is $\tilde{\Phi}(f) \in \mathcal{L}(k)$ with

$$\langle \tilde{\Phi}(f)x, y \rangle = \int_{\hat{A}} f dE_{x,y}.$$

We now analyze the properties of the

map $\underline{\Phi} : \mathcal{B}^*(\hat{A}) \rightarrow \mathcal{L}(X)$.

Clearly $\underline{\Phi}$ is linear.

$$\text{First, from } \langle Tx, y \rangle = \int_{\hat{A}}^T dE_{x,y}$$

$$= \langle \underline{\Phi}(T)x, y \rangle$$

for $T \in A$, $x, y \in X$ follows that

$$\underline{\Phi}(\hat{T}) = T$$

and thus $\underline{\Phi}$ extends the inverse of the
Gelfand transform.

Next we compute:

$$\langle \underline{\Phi}(f)^* x, y \rangle = \langle x, \underline{\Phi}(f) y \rangle = \langle \underline{\Phi}(f)x, y \rangle$$

$$= \overline{\int_{\hat{A}} f dE_{x,y}} = \int_{\hat{A}} \bar{f} dE_{x,y} \quad \text{since}$$

$E_{x,y}$ is a positive measure. The

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integral equals $\langle \bar{\Phi}(f) x, x \rangle$ and
thus $\langle \bar{\Phi}(f)^* x, x \rangle = \langle \bar{\Phi}(f)x, x \rangle \quad \forall x$
 $\Rightarrow \bar{\Phi}(f)^* = \bar{\Phi}(f) \quad \forall f \in \mathcal{B}^\infty(\hat{A})$.

Now we show that $\bar{\Phi}(fg) = \bar{\Phi}(f)\bar{\Phi}(g)$

$\forall f, g \in \mathcal{B}^\infty(\hat{A})$.

Let $s, t \in A$, then $(\widehat{st}) = \widehat{s} \widehat{t}$
hence

$$\begin{aligned} \int_{\hat{A}} \widehat{s} \widehat{t} dE_{x,y} &= \langle st(x), y \rangle \\ &= \int_{\hat{A}} \widehat{s} dE_{T(x), y} \end{aligned}$$

Now if two complex measures coincide
on $C(\hat{A})$ they coincide on $\mathcal{B}^\infty(\hat{A})$
and hence we can replace \widehat{s} by any
 $f \in \mathcal{B}^\infty(\hat{A})$.

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More precisely, the complex measure

$\hat{T} dE_{x,y}$ and $d\bar{E}_{T(x),y}$ coincide

on $C(\hat{A})$ hence on $\mathcal{B}^\infty(\hat{A})$.

For every $f \in \mathcal{B}^\infty(\hat{A})$ we get then:

$$\int_{\hat{A}} f \hat{T} dE_{x,y} = \int_{\hat{A}} f d\bar{E}_{T(x),y}$$

$$= \langle \hat{\varphi}(f) T(x), y \rangle = \langle T(x), z \rangle$$

where $z = \hat{\varphi}(f) y$ and the latter

equals

$$= \int_{\hat{A}} \hat{T} dE_{x,z}$$

Thus the complex measures $f \cdot dE_{x,y}$

and $d\bar{E}_{x,z}$ coincide on $C(\hat{A})$

hence on $\mathcal{B}^-(\hat{A})$.

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That is: $\int \hat{A} f \cdot g dE_{x,y} = \int \hat{A} g dE_{x,z}$

This implies:

$$\begin{aligned}\langle \hat{A}(f \cdot g)_{x,y} \rangle &= \int \hat{A} f \cdot g dE_{x,y} \\ &= \int \hat{A} g dE_{x,z} \\ &= \langle \hat{A}(g)_{x,z} \rangle = \langle \hat{A}(g)_{x,z}, \hat{A}(f)_{y,z} \rangle \\ &= \cancel{\langle \hat{A}(g) \hat{A} \rangle} = \langle \hat{A}(f) \hat{A}(g)_{x,y} \rangle\end{aligned}$$

Which finally shows that

$$\hat{A}(f \cdot g) = \hat{A}(f) \hat{A}(g) \quad \forall f, g \in S^0(\hat{A})$$

Now we can define E : if $w \in \hat{A}$

is a Borel subset, we define

$$E(w) := \hat{A}(x_w)$$

where as usual, χ_ω is the characteristic function of ω .

$$1) E(\phi) = \overline{\Phi}(\phi) = 0;$$

$$E(\hat{x}) = \overline{\Phi}(1) \quad \text{and}$$

$$\begin{aligned} \langle \overline{\Phi}(1)x, \frac{x}{\|x\|} \rangle &= \int_{\hat{A}} dE_{x, \frac{x}{\|x\|} x} = \|x\|^2 \\ &= \langle x, x \rangle \end{aligned}$$

$$\text{and hence } \overline{\Phi}(1) = \text{Id}_{\mathcal{H}}.$$

$$2) \text{ Since } X_\omega = \overline{X}_\omega \text{ and } \overline{X}_\omega^* = X_\omega,$$

and $\overline{\Phi}$ is a $*$ -homomorphism we deduce that $E(\omega)$ is a self-adjoint projection.

$$\begin{aligned} 3) E(\omega \cap \omega') &= \overline{\Phi}(X_{\omega \cap \omega'}) = \overline{\Phi}(X_\omega \cdot X_{\omega'}) \\ &= \overline{\Phi}(X_\omega) \overline{\Phi}(X_{\omega'}) \\ &= E(\omega) E(\omega'). \end{aligned}$$

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4.) If $\omega \cap \omega' = \emptyset$ then

$$\bar{E}_{x,x}(\omega \cup \omega') = \bar{E}_{x,x}(\omega) + \bar{E}_{x,x}(\omega')$$

that is $\langle \bar{E}(\omega \cup \omega')_{x,x} \rangle =$

$$= \langle (\bar{E}(\omega) + \bar{E}(\omega'))_{x,x} \rangle$$

from which $\bar{E}(\omega \cup \omega') = \bar{E}(\omega) + \bar{E}(\omega')$

follows.

5.) $E_{x,x}(\omega) = \langle E(\omega)_{x,x} \rangle$ is a

regular positive Borel measure by construction.

It follows now that ~~\bar{E} is a measure~~ from

$$\langle \bar{E}(f)_{x,y} \rangle = \int \hat{A} f dE_{x,y}$$

that \bar{E} factors via the projection

$B^\infty(\hat{A}) \rightarrow L^\infty(E)$ and induces

the map $\gamma: L^\infty(E) \rightarrow \mathcal{L}(\mathbb{A})$ from

Thm. 5.16. Hence $\|\Phi(f)\| = \|f\|_\infty$.

This shows assertions (1) and (2)
from the theorem.

Now (3) follows since every $f \in L^\infty(E)$
is a uniform limit of simple functions.

(4). If $\omega \subset \hat{\mathbb{A}}$ is open and

$E(\omega) = 0$ then $E_{x_1, x_2}|_\omega$ is the

zero measure $\forall x \in \mathbb{A}$. Let $T \in A$

with $\text{supp}(\hat{T}) \subset \omega$. Then

$$\langle T_{x_1, x_2}, \rangle = \int \hat{T} dE_{x_1, x_2} = 0 \quad \forall x \in \mathbb{A}$$

hence $T = 0$. But this implies $\omega = \emptyset$.

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(5) Let $S \in \mathcal{L}(\mathbb{H})$, $T \in A$

then:

$$\langle STx, y \rangle = \langle Tx, S^*y \rangle = \int \hat{T} dE_{x, S^*y}$$

$$\langle TSx, y \rangle = \int \hat{T} dE_{Sx, y}$$

Then

$$\langle STx, y \rangle = \langle TSx, y \rangle \quad \forall x, y \in \mathbb{H}$$

$$\iff dE_{x, S^*y} = dE_{Sx, y} \quad \forall x, y$$

$$\iff E_{x, S^*y}(\omega) = E_{Sx, y}(\omega) \quad \forall x, y$$

$$\iff \langle E(\omega)x, S^*y \rangle = \langle E(\omega)Sx, y \rangle$$

and since $\langle E(\omega)x, S^*y \rangle = \langle S^*E(\omega)x, y \rangle$

this implies (5). □