

If we want to apply the spectral theorem to a single normal operator $T \in \mathcal{L}(\mathbb{K})$ we will use Thm 4.10 and the construction preceding it. Namely we consider

$$\mathcal{B} := \overline{\left\{ P(T, T^*) : \rho \in \mathbb{C}[x, t] \right\}}$$

the abelian sub C^* -algebra generated by T, T^*, Id . and recall that here $\text{Sp}(T)$ refers to $\text{Sp}_{\mathcal{L}(\mathbb{K})}(T) = \text{Sp}_{\mathcal{B}}(T)$.

The map (Thm 4.10)

$$\begin{aligned} \phi : \hat{\mathcal{B}} &\longrightarrow \text{Sp}(T) \\ x &\longmapsto x(T) \end{aligned}$$

is then a homeomorphism.

Furthermore we have shown that

if ~~given~~ given $f \in C(S_p(T))$,
 $f(T)$ denotes the unique element in
 B such that $\widehat{f(T)}(x) = f(x(T))$ (*)

then the resulting map

$$C(S_p(T)) \longrightarrow B$$

$$f \mapsto f(T)$$

is a C^* -algebra isomorphism.

Observe that the identity (*) can also be written:

$$\widehat{f(T)}(ev^{-1}(\lambda)) = f(\lambda) \quad \forall \lambda \in S_p(T)$$

The spectral theorem says then that there is a unique resolution E of the identity s.t.

$$b = \int \widehat{b} dE \quad \forall b \in B.$$

Define then $\forall \omega \in \text{Sp}(\tau)$ Borel set

$$E'(\omega) := E(ev'(\omega)) \quad (+)$$

Then one verifies that E' is a resolution of identity on $\text{Sp}(\tau)$. Observe that

since $ev : \hat{B} \rightarrow \text{Sp}(\tau)$ is continuous,

we can define for every positive regular

Borel measure μ on \hat{B} its pushforward

$$(ev_*)(\mu)(\omega) := \mu(ev^{-1}(\omega))$$

positive

which produces a regular Borel

measure $ev_*(\mu)$ on $\text{Sp}(\tau)$. Using

that ev is a homeomorphism one

checks: $\forall f \in C(\hat{B})$:

$$\int_{\hat{B}} f(x) d\mu(x) = \int_{\text{Sp}(\tau)} (f \circ ev')(y) d(ev_*\mu)(y)$$

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With these notations one deduces from

$$(*) \text{ that } E'_{x_1 x_2} = (\text{ev})^* E_{x_1 x_2} \quad \forall x \in \mathcal{H}.$$

We have then: $\forall b \in V$

$$\begin{aligned} \langle bx, x \rangle &= \int_{\hat{B}} \hat{b}(x) dE_{x_1 x_2}(x) \\ &= \int \left(\hat{b} \circ \text{ev}^{-1} \right)(y) d(\text{ev}_* E_{x_1 x_2})(y) \\ &= S_p(\tau) \end{aligned}$$

which leads with $b = f(\tau)$ to .

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$$\begin{aligned} \langle f(\tau)_{\infty, \infty} \rangle &= \int (\widehat{f(\tau)} e^{-i\lambda}) (\lambda) d(e^{\lambda} E_{\infty, \infty})(\lambda) \\ &= \int_{\mathbb{R}} f(\lambda) dE'(\lambda), \\ &\quad S_p(\tau) \end{aligned}$$

Thus we conclude from the spectral theorem that there is a unique resolution of identity E' on $S_p(\tau)$ such that

$$f(\tau) = \int_{S_p(\tau)} f(\lambda) dE'(\lambda) \quad \forall f \in C(S_p(\tau)).$$

It is important to realize that $f(\tau)$ takes a very concrete form if $f(\lambda) = p(\lambda, \bar{\lambda})$ where $p \in \mathbb{C}[X, Y]$ is a polynomial. Indeed since $f \mapsto f(\tau)$

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$\exists \in C^*$ -isomorphism sending id to T ,
 $\mathbb{1}$ to Id , and hence $\overline{\text{id}}$ to T^* ,

we get that for $f(\lambda) = p(\lambda, \bar{\lambda})$,

$f(T) = p(T, T^*)$. Thus for

any $p \in \mathbb{C}[x, y]$:

$$p(T, T^*) = \int_{\mathbb{R}} p(\lambda, \bar{\lambda}) dE'(\lambda).$$

$S_p(T)$

If now E'' is a resolution of identity

on $S_p(T)$ such that

$$T = \int \lambda dE''(\lambda) \quad (*)$$

$S_p(T)$

then it follows from ~~(*)~~ that the

C^* -algebra map $\gamma: L^\infty(E'') \rightarrow \mathcal{A}^{(1)}$

from Thm 5.16 sends id to T

hence $\overline{\text{id}}$ to T^*

and hence as a result

$$p(T, T^*) = \int_{\text{Sp}(T)} p(\lambda, \bar{\lambda}) dE''(\lambda)$$

and hence since the functions of the form $\lambda \mapsto p(\lambda, \bar{\lambda})$ are dense in $C(\text{Sp}(T))$

we get

$$f(T) = \int_{\text{Sp}(T)} f(\lambda) dE''(\lambda) \quad \forall f \in C(\text{Sp}(T))$$

which implies $E'' = E'$.

This proves essentially:

Corollary 5.20 Let $T \in L(H)$ be normal

and $\text{Sp}(T) \subset \mathbb{C}$ its spectrum. There is a unique resolution of identity on $\text{Sp}(T)$ which satisfies

$$T = \int_{\text{Sp}(T)} \lambda dE(\lambda).$$

Moreover if $S \in L(H)$ commutes with T and it commutes with every projection $E(\omega)$.

Remark: One can show that if $T \in \mathcal{L}(\mathfrak{X})$ is normal and f commutes with T then it commutes with T^* .

Next, if E is the resolution of identity on $\mathfrak{S}_p(T)$ we have seen that $\forall f \in C(\mathfrak{S}_p(T))$

$$f(T) = \int_{\mathfrak{S}_p(T)} f(\lambda) dE(\lambda).$$

The map $B^\infty(\mathfrak{S}_p(T)) \xrightarrow{\text{by}} L^\infty(E) \xrightarrow{\text{by}} \mathcal{L}(\mathfrak{X})$

given by Thm 5.16 is related to E

by $\underline{\text{by}}(f) = \int_{\mathfrak{S}_p(T)} f(\lambda) dE(\lambda)$

hence extends $C(\mathfrak{S}_p(T)) \rightarrow B$
 $f \rightarrow f(T)$

and we will thus for $f \in B^\infty(\mathfrak{S}_p(T))$

denote $\text{ht}(f)$ by $f(T)$. Thus for normal operators we have extended the continuous functional calculus to the bounded Borel functional calculus, and:

Corollary 5.21 The map $\mathcal{B}^\infty(\sigma_p(T)) \rightarrow \mathcal{L}(X)$
 $f \mapsto f(T)$

is a C^* -homomorphism sending 1 to

$I_{\mathcal{A}_X}$, id to T and

$$\|f(T)\| \leq \|f\| = \sup\{|f(\lambda)| : \lambda \in \sigma_p(T)\}.$$

If $f \in C(\sigma_p(T))$ then $\|f(T)\| = \|f\|$.

$$\text{Moreover: } \|f(T)x\|^2 = \int |f(\lambda)|^2 dE_{x,x} \quad \text{over } \sigma_p(T)$$

and T is limit in $\|\cdot\|$ -topology of
finite linear combinations of projections $E(w)$.

Schur's Lemma

An important application ~~to~~ of the spectral theorem is to an irreducibility criterion of unitary representations called Schur's lemma. First a little terminology:

given a group G , a unitary representation of G in the Hilbert space \mathcal{H} is a group homomorphism $\pi: G \rightarrow U(k)$ of G into the group $U(\mathcal{H})$ of unitary operators. If $\mathcal{L} \subset \mathcal{H}$ is an invariant subspace, that is, $\pi(g)\mathcal{L} \subset \mathcal{L} \forall g \in G$ then so is \mathcal{L}^\perp .

Def 5.22 The representation (π, \mathcal{H}) is irreducible if whenever \mathcal{L} is a closed invariant subspace we have either

$$\mathcal{L} = \{0\} \text{ or } \mathcal{L} = \mathcal{H}.$$

In this context the following plays a fundamental role:

$$\text{Int}(\pi) = \left\{ T \in \mathcal{L}(X) : T\bar{\pi}(y) = \pi(y)T \forall y \in G \right\}.$$

Then $\text{Int}(\pi)$ is a C^* -subalgebra of

$$\mathcal{L}(X) \text{ containing } \left\{ \lambda \text{Id}_X : \lambda \in \mathbb{C} \right\}.$$

Observe for instance that if $\mathcal{X} \subset X$

is a closed invariant subspace and

$\underline{P} : X \rightarrow \mathcal{L}$ the orthogonal projection,

then $\underline{P} \in \text{Int}(\pi)$.

Thm 5.23 The representation (π, X) is

irreducible $\iff \text{Int}(\pi) = \left\{ \lambda \text{Id}_X : \lambda \in \mathbb{C} \right\}$