

4. Spectrum in C^* -algebras and the Gelfand isomorphism for abelian C^* -algebras

We recall that an involutive Banach algebra A is one admitting an ~~involution~~ ^{map} $x \mapsto x^*$, satisfying $(x^*)^* = x$, $x \mapsto x^*$ is \mathbb{C} -antilinear, $(xy)^* = y^*x^*$ and $\|x^*\| = \|x\|$.

If in addition $\|x^*x\| = \|x^*\| \cdot \|x\| \quad \forall x \in A$ it is a C^* -algebra.

The main objective of this section is to show that for abelian C^* -algebras the Gelfand transform is an isomorphism.

First we establish some basic facts about C^* -algebras without assuming

that they are abelian.

Let A be an involutive Banach algebra.

Def. 4.1 An element $x \in A$ is self-adjoint

if $x = x^*$, normal if $xx^* = x^*x$

and if A has a unit e is unitary if

$$xx^* = x^*x = e.$$

Clearly self-adjoint and unitary elements

are normal. Also every x can be written

$$\text{as } x = x_1 + i x_2$$

where x_1, x_2 are self-adjoint:

$$x_1 = \frac{1}{2}(x + x^*), \quad x_2 = \frac{1}{2i}(x - x^*).$$

The following fact about C^* -algebras
is fundamental:

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Prop. 4.2 Let A be a C^* -algebra.

If x is normal then $\|x\| = \|x\|_{op}$.

Proof:

Let $n \geq 1$ and x normal. We compute

$$\begin{aligned}\|x^{2^n}\|^2 &= \|(x^{2^n})^* x^{2^n}\| = \|(x^* x)^{2^n}\| \\ &= \|(x^* x)^n \cdot (x^* x)^{n-1}\| = \|(x^* x)^n\|^2.\end{aligned}$$

Now we apply this to:

$$\begin{aligned}\|x^{2^n}\| &= \|(x^* x)^{2^{n-1}}\| = \|(x^* x)^{2^{n-1}} (x^* x)^{2^{n-2}}\| \\ &= \|(x^* x)^{2^{n-2}}\|^2 = \|(x^* x)^{2^{n-3}}\|^2 = \\ &= \dots = \|(x^* x)\|^{2^n} = \|x\|^{2^n}\end{aligned}$$

Thus $\|x\| = \|x^{2^n}\|^{1/2^n} \longrightarrow \|x\|_{op}$.



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The next problem we face is that even if A is a C^* -algebra, if we put the usual norm $\|(x, \lambda)\| = \|x\| + |\lambda|$ on its unitalisation $A_{\mathbb{I}}$, then it is not a C^* -norm.

Indeed:

$$\|(x, \lambda)^*(x, \lambda)\| = \|x^*x + \lambda x^* + \bar{\lambda}x\| + |\lambda|^2$$

and

$$\|(x, \lambda)\|^2 = ((\|x\| + |\lambda|)^2) = \|x\|^2 + 2(\|x\||\lambda| + |\lambda|)$$

Equality would imply:

$$2\|x\||\lambda| \leq \|\lambda x^* + \bar{\lambda}x\|$$

in particular for $\lambda = i$:

$$2\|x\| \leq \|x^* - x\|$$

which would imply $x = 0$ for every self-adjoint x .

Prop. 4.3 If A is not a unital C^* -algebra

then there is a norm on $A_{\mathbb{I}}$ making it
a C^* -algebra, extending the C^* -algebra
norm on A .

Proof.

Let as usual $A_{\mathbb{I}} = A \times \mathbb{C}$ and observe

that A is an ideal in $A_{\mathbb{I}}$. We can

thus consider for every $x \in A_{\mathbb{I}}$ the

left multiplication map

$$\begin{aligned} L_x : A &\longrightarrow A \\ y &\mapsto x \cdot y. \end{aligned}$$

We obtain in this way a map

$$A_{\mathbb{I}} \longrightarrow \mathcal{L}(A), x \mapsto L_x$$

which is certainly an algebra homomorphism.

Observe first that for $x \in A$

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$$\|L_x\| = \sup_{\|y\| \leq 1} \|x \cdot y\| \leq \|x\| \quad \text{and}$$

since $\|xx^*\| = \|x\|^2$ setting

$$y = \frac{x^*}{\|x\|}$$

we obtain $\|y\|=1$ and $\|L_x(y)\| = \|x\|$

which implies $\|x\| = \|L_x\| \quad \forall x \in A$.

Define now for $x \in A_I$:

$$\|x\| := \|L_x\|.$$

It is clear that all the properties of a Banach algebra norm are fulfilled

except for one: we have to show that if $\|x\| = 0$ then $x = 0$, in other words

if $L_x = 0$ then $x = 0$. Now let

$x = x' + \lambda e$, $\lambda \neq 0$ be such that

$$x \cdot y = 0 \quad \forall y \in A.$$

Then : $x'y + \lambda y = 0$, that is,

$$\left(-\frac{1}{\lambda}\right)x'y = y$$

so that $\left(-\frac{1}{\lambda}\right)x'$ is a left identity

which implies that $\left(-\frac{1}{\lambda}x'\right)^*$ is a right

identity. But then:

$$\left(-\frac{1}{\lambda}\right)x' = \left(-\frac{1}{\lambda}x'\right)\left(-\frac{1}{\lambda}x'\right)^* = \left(-\frac{1}{\lambda}x'\right)^*$$

showing that $\left(-\frac{1}{\lambda}x'\right)$ is an identity of
 A , contradiction.

Finally we have to verify that " "

is a C^* -algebra norm on $A_{\mathbb{I}}$.

Let $x \in A_{\mathbb{I}}$, $a \in A$; since $x a \in A$

$$\begin{aligned} \|L_x(a)\|^2 &= \|xa\|^2 = \|\left(xa\right)^*(xa)\| = \|a^*x^*xa\| \\ &\leq \|a^*\| \|L_{x^*x}(a)\| \leq \|a\|^2 \|L_{x^*x}\| \end{aligned}$$

Taking the sup over all $a \in A$ with

$\|a\| \leq 1$ we get:

$$\|L_x\|^2 \leq \|L_{x^*x}\| \text{ hence } \|x\|^2 \leq \|x^*x\|.$$

But $\|x^*x\| \leq \|x^*\| \|x\|$ which implies

$$\|x\| \leq \|x^*\| \quad \forall x \in A_I$$

$$\text{hence } \|x\| = \|x^*\| \quad \forall x \in A_I$$

$$\text{Using again } \|x\|^2 \leq \|x^*x\|$$

$$\text{and } \|x^*x\| \leq \|x^*\| \|x\| = \|x\|^2$$

$$\text{We get } \|x^*x\| = \|x\|^2. \quad \boxed{\beta}$$

The next proposition establishes natural properties for the spectrum of unitary and self-adjoint elements in a unital C^* -algebra; Prop. 4.3 is then crucial to