

## 1. Banach Algebras: Definitions and Examples.

We begin recalling a central notion from Algebra I:

Df. 1.1 A  $\mathbb{C}$ -algebra (short: algebra) is a  $\mathbb{C}$ -vector space  $A$  endowed with a bilinear map

$$A \times A \rightarrow A$$

$$(x, y) \mapsto x \cdot y$$

called product (or multiplication) and

such that  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$$\forall x, y, z \in A.$$

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It is called unital if there exists  
 $e \in A$ , ~~such~~ such that:

$$e \cdot x = x \cdot e = x \quad \forall x \in A.$$

It is called abelian (or commutative) if  
 $x \cdot y = y \cdot x \quad \forall x, y \in A.$

Def. 1.2. An Ideal in  $A$  is a  $\mathbb{C}$ -  
vector subspace  $I$  such that:

$$x \cdot I \subset I, I \cdot x \subset I \quad \forall x \in A.$$

It is then a standard fact that the  
multiplication on  $A$  descends to a  
well defined multiplication on  $A/I$ .

Construction 1.3 Any algebra  $A$  embeds  
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as follows:  $A_{\mathbb{I}} := A \times \mathbb{C}$  as

$\mathbb{C}$ -vector space. Define:

$$(x, z)(y, \mu) = (xy + \bar{z}y + \mu x, z \cdot \mu)$$

Then this defines an algebra structure

on  $A_{\mathbb{I}}$  with unit  $e := (0, 1)$  into

which  $A$  embeds:  $A \rightarrow A_{\mathbb{I}}$   
 $x \mapsto (x, 0)$

as an ideal.

The central definition of this course

is

Def. 1.4 A Banach Algebra is a  
 $\mathbb{C}$ -algebra  $A$  endowed with a  
norm  $\| \cdot \|$  for which  $A$  is a Banach  
space and satisfying  $\| xy \| \leq \| x \| \| y \|$   
 $\forall x, y \in A$ .

Construction 1.5: If  $A$  is a Banach

algebra, define on  $A_{\mathbb{I}} = A \times \mathbb{C}$ :

$$\|(x, \lambda)\| = \|x\| + |\lambda|.$$

Then  $A_{\mathbb{I}}$  becomes a unital Banach algebra.

Remark 1.6. (Exercise)

If  $A$  is a unital  $\mathbb{C}$ -algebra with a Banach space norm  $\|\cdot\|$  for which  $A \times \mathbb{C} \rightarrow A$  is continuous then there is an equivalent norm  $\|\cdot\|'$  s.t.  $(A, \|\cdot\|')$  is a (unital) Banach algebra and  $\|e'\| = 1$ .

Def. 1.7 If a Banach Algebra  $A$  admits

a map  $x \mapsto x^*$  with the following

properties (i)  $(x^*)^* = x$

(ii)  $(x+y)^* = x^*+y^*$

(iii)  $(\alpha x)^* = \bar{\alpha} x^*$

(iv)  $(x \cdot y)^* = y^* \cdot x^*$

(v)  $\|x^*\| = \|x\|$

for every  $x, y \in A$ ,  $\alpha \in \mathbb{C}$  then  $A$  is

called an involutive Banach algebra and

the map:  $x \mapsto x^*$  the involution on  $A$ .

If the involution satisfies the additional conditions:

$$(VI) \|x^* \cdot x\| = \|x^*\| \cdot \|x\|$$

$$\forall x \in A$$

then  $A$  is called a  $C^*$ -algebra.

Exercise 1.8 : If  $e$  is a unit in an involutive Banach algebra then  $e^* = e$ .

There are three main classes of Banach algebras which can roughly be described as

- algebras of functions with pointwise multiplication
- algebras of operators with composition of operators
- group algebras with convolution product.

We now give examples of Banach algebras in each of these three classes.

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Example 1.9 (1) (commutative  $C^*$ -algebra)

Let  $X$  be locally compact Hausdorff.

The  $\mathbb{C}$ -vector space

$$C^b(X) := \left\{ f: X \rightarrow \mathbb{C} : f \text{ is continuous and bounded} \right\}$$

is a  $\mathbb{C}$ -algebra for pointwise multi-

plication:  $(f \cdot g)(x) := f(x) \cdot g(x) \quad \forall x \in X$

The norm  $\|f\|_\infty := \sup_{x \in X} |f(x)|$

makes it a commutative unital Banach

algebra. Let  $f^*(x) := \overline{f(x)}$ ;

then  $f \mapsto f^*$  is an involution and

$$\|f^* \cdot f\|_\infty = \sup_{x \in X} |\overline{f(x)} \cdot f(x)|$$

$$= \sup_{x \in X} |f(x)|^2 = \|f\|^2 = \|f^*\| \|f\|$$

Thus it is a  $C^*$ -algebra.

Recall that  $f: X \rightarrow \mathbb{C}$  vanishes at infinity if  $\forall \varepsilon > 0 \exists K \subset X$  compact such that

$$|f(x)| < \varepsilon \quad \forall x \in X \setminus K.$$

The space

$C_0(X) = \{f: X \rightarrow \mathbb{C} : f \text{ is continuous and vanishes at infinity}\}$

endowed with  $\|\cdot\|_\infty$  is a commutative  $C^*$ -algebra; it is unital iff  $X$  is compact.

In fact as a consequence of Gelfand's theory of commutative Banach algebras we will see that any commutative  $C^*$ -algebra is isomorphic to  ~~$C_0(X)$~~   $C_0(X)$  for an appropriate  $X$ .

(2)  $X \subset \mathbb{C}$  compact subset :

Let  $A(X) := \{ f: X \rightarrow \mathbb{C} : f \text{ is continuous and holomorphic in the interior } \overset{\circ}{X} \text{ of } X \}$ . Then with  $\|f\|_\infty$  it is a ~~unitary~~ unital Banach subalgebra of  $C(X)$ . Observe that

if  $\overset{\circ}{X} \neq \emptyset$  then  $f^*(z) := \overline{f(z)}$  is not an involution because  $f^*$  is not holomorphic in  $\overset{\circ}{X}$ , in general.

(3) Let  $a < b$  and  $n \in \mathbb{N}$ :

let  $C^n([a, b]) = \{ f: [a, b] \rightarrow \mathbb{C}$   $n$ -times continuously differentiable with pointwise multiplication and  $\|f\| := \sum_{k=0}^n \frac{1}{k!} \|f^{(k)}\|_\infty$

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Then it follows from Analysis I that

$C^n([a,b])$  is a Banach space. In

addition,

$$\|f \cdot g\| = \sum_{k=0}^n \frac{1}{k!} \|(f \cdot g)^{(k)}\|_\infty$$

$$= \sum_{k=0}^n \frac{1}{k!} \left\| \sum_{j=0}^k \binom{k}{j} f^{(j)} \cdot g^{(k-j)} \right\|_\infty$$

$$= \sum_{k=0}^n \left\| \sum_{j=0}^k \frac{1}{j!(k-j)!} f^{(j)} g^{(k-j)} \right\|_\infty$$

$$\leq \sum_{k=0}^n \sum_{j=0}^k \frac{1}{j!} \|f^{(j)}\|_\infty \frac{1}{(k-j)!} \|g^{(k-j)}\|_\infty$$

$$\leq \sum_{l=0}^n \sum_{s=0}^l \frac{1}{l!} \|f^{(l)}\|_\infty \frac{1}{s!} \|g^{(s)}\|_\infty$$

$$= \|f\| \cdot \|g\|.$$

Thus  $C^n([a,b])$  is a commutative unital Banach algebra.

It is a non-trivial fact we'll show later  
that  $C^\infty([a, b])$  does not admit  
any Banach algebra norm.

#### (4) (Volterra Algebra)

Let  $\lambda$  be the Lebesgue measure on  $\mathbb{R}$

restricted to  $[0, 1]$  and  $L^1([0, 1])$

the space of measurable absolutely

integrable functions on  $[0, 1]$  with

norm  $\|f\|_1 = \int_0^1 |f(x)| d\lambda(x).$

It is a classical fact that  $L^1([0, 1])$

is a Banach space. Using Fubini

one can show that if  $f, g \in L^1([0, 1])$ ,

$$f * g(x) = \int_0^x f(x-t) g(t) d\lambda(t)$$

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exists for a.e.  $x \in [0,1]$  and  
belongs to  $L^1([0,1])$ .

The operation  $f * g$  is obviously  
bilinear. By the substitution  $t \mapsto x-t$

we see that it is commutative. Next:

$$\begin{aligned} \|f * g\|_1 &= \int_0^1 \left( \int_0^\infty f(x-t)g(t) d\lambda(t) \right) d\lambda(x) \\ &\leq \int_0^1 \left[ \int_0^\infty |f(x-t)| |g(t)| d\lambda(t) \right] d\lambda(x) \\ &= \int_0^1 \underbrace{\int_t^1 |f(x-t)| d\lambda(x)}_{\int_0^x |f(y)| d\lambda(y)} |g(t)| d\lambda(t) \\ &\leq \underbrace{\int_0^x |f(y)| d\lambda(y)}_{\|f\|_1} \cdot \underbrace{\int_0^1 |g(t)| d\lambda(t)}_{\|g\|_1} \\ &\leq \|f\|_1 \cdot \|g\|_1. \end{aligned}$$

Thus  $L^1([0,1])$  is a Banach algebra  
that is commutative. It is an interesting  
exercise to show that  $L'([0,1])$  has no  
unit.

(5) Let  $B$  be a Banach space with  
norm  $\|\cdot\|$  and  $\mathcal{L}(B) := \{T: B \rightarrow B$   
linear continuous $\}$ . Then  $\mathcal{L}(B)$   
is a  $C$ -algebra for the composition.

It is a fact from FAI that  
 $\mathcal{L}(B)$  is characterized as the span of  
linear maps  $T: B \rightarrow B$  for which

$$\|T\| := \sup_{\|x\| \leq 1} \|T(x)\| < +\infty.$$

In FAI it was shown that

$$\|\bar{T}_1 T_2\| \leq \|T_1\| \cdot \|T_2\|$$

And thus  $\mathcal{L}(\mathbb{B})$  is a unital Banach algebra; it is not commutative, unless  $\dim \mathbb{B} = 1$ .

(6) Let now  $\mathcal{H}$  be a Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$ . Recall that the adjoint  $T^*$  of  $T \in \mathcal{L}(\mathcal{H})$  is uniquely defined by

$$\langle T^*v, w \rangle = \langle v, Tw \rangle \quad \forall v, w \in \mathcal{H}.$$

Then  $(T^*)^* = T$  and properties (i), (ii), (iii), (iv) of Def. 1.7 are easily verified. Concerning (v) and (vi):

$\forall v \in \mathcal{H}$ :

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \langle T^*T v, v \rangle$$

$$\leq \|T^*T v\| \|v\| \leq \|T^*T\| \|v\|^2$$

which implies  $\|T\|^2 \leq \|T^*T\|$  ;

together with  $\|T^*\| \leq \|T^*\| \cdot \|T\|$

this implies  $\|T\| \leq \|T^*\|$  and

hence  $\|T\| \leq \|T^*\| \leq \|(T^*)^*\| = \|T\|$

which shows  $\|T\| = \|T^*\|$  and

hence (v) ; for (vi) we come back to

$$\|T\|^2 \leq \|T^*T\|$$

and deduce from  $\|T^*T\| \leq \|T^*\| \cdot \|T\|$

and  $\|T^*\| = \|T\|$  that :

$$\|T^*T\| = \|T^*\| \|T\| .$$

Hence  $\mathcal{L}(\mathfrak{A})$  with the operation  $T \mapsto T^*$

is a unital  $C^*$ -algebra.

In particular Gelfand's structure theorem  
for commutative  $C^*$ -algebras can be

applied to commutative sub- $C^*$ -algebra  
of  $\mathcal{L}(\mathfrak{H})$  leading to spectral theorems  
for unitary, self-adjoint or more gene-  
rally normal operators.

(7) Let  $\Gamma$  be a group, for instance  
countable. Then

$$\ell^1(\Gamma) := \left\{ f: \Gamma \rightarrow \mathbb{C} : \sum_{g \in \Gamma} |f(g)| < +\infty \right\}$$

is a Banach space with norm

$$\|f\|_1 := \sum_{r \in \Gamma} |f(r)|.$$

Define  $\delta_\gamma: \Gamma \rightarrow \mathbb{C}$  as the character-  
istic function of  $\{\gamma\}$ . In algebra  
one introduces  $\mathbb{C}[\Gamma]$  "the group algebra",  
that is the  $\mathbb{C}$ -vector space generated by

$\{\delta_\gamma : \gamma \in \Gamma\}$  with a product

that is the linear extension of:

$$\delta_\gamma * \delta_\delta := \delta_{\gamma\delta}.$$

Clearly  $\mathbb{C}[\Gamma] \subset \ell^1(\Gamma)$  and this product extends to a product on  $\ell^1(\Gamma)$

given by :

$$f * g(\gamma) = \sum_{\gamma \in \Gamma} f(\gamma_2)g(\gamma')$$

called the convolution product.

There is a natural involution on  $\ell^1(\Gamma)$

defined by  $f^*(\gamma) := \overline{f(\gamma T^{-1})}$

which makes  $\ell^1(\Gamma)$  an involutive

Banach algebra. It has an identity

namely  $\delta_e$  and it is commutative iff  $\Gamma$  is.