

For a complex measure μ on \widehat{G} let us denote $\tilde{\mu}(x) := \int_{\widehat{G}} f(x, x) d\mu(x)$.

Let $P(G) = \{f: G \rightarrow \mathbb{C} \text{ is continuous and positive definite}\}$

then $P(G)$ is a convex cone; let $B(G)$ be the \mathbb{C} -vector space generated by $P(G)$; it can be seen as a \mathbb{C} -vector subspace of the space $C_b(G)$ of all continuous bounded functions. This space will play a key role in the inversion theorem. For the moment we can draw the following conclusion from Banach's theorem:

Corollary 9.7 The map $\mu \mapsto \tilde{\mu}$ defines
a \mathbb{C} -linear bijection:

$$M(\hat{G}) \rightarrow B(G).$$

Proof.

The injectivity is the same argument than
the uniqueness statement in Bochner's thm.

and left to the reader as exercise.

That the map takes its values in $B(G)$
follows from the fact that every complex
measure $\mu = \mu_1 + i\mu_2$ can be written

$$\text{as } \mu = (\mu_1^+ - \mu_1^-) + i(\mu_2^+ - \mu_2^-)$$

as a combination of 4 bounded positive
measures.

That the map is surjective follows from
the definition of $B(G)$ and Bochner's thm.

9.2. The inversion theorem.

Theorem 9.8 Given a Haar measure λ on G there is a Haar measure ω on \widehat{G} such that

(1) If $f \in L^1(G) \cap B(G)$ then $\hat{f} \in L^1(\widehat{G})$.

(2) For every $f \in L^1(G) \cap B(G)$:

$$f(x) = \int_{\widehat{G}} \hat{f}(x)(\omega, x) d\omega(x) \quad \forall x \in G.$$

Given $f \in B(G)$ let $\mu_f \in M(\widehat{G})$ be the complex measure (see Cor. 8-7) such that $f = \widehat{\mu}_f$. The main point is to show that if we fix a Haar measure λ on G there is a Haar measure

ω on \widehat{G} such that

$$d\mu_f = \widehat{f} \cdot d\omega \quad \forall f \in L^1(G) \cap B(C)$$

Lemma = 9.9. $\forall f, g \in L^1(G) \cap B(C)$

We have

$$\widehat{g} \cdot d\mu_f = \widehat{f} \cdot d\mu_g.$$

Proof: $\forall h \in L^1(G)$ and $f \in L^1(G) \cap B(C)$:

$$h * f(x) = \int_G h(x) f(x') d\lambda(x)$$

$$= \int_G h(x) \int_{\widehat{G}} (\widehat{x}, x) d\mu_f(x)$$

$$= \int_{\widehat{G}} \widehat{h}(x) d\mu_f(x). \quad (*)$$

If $g \in L^1(G) \cap B(C)$ then replacing
 h by $h * g$ we get:

$$\int_{\widehat{G}} \widehat{h} \widehat{g} d\mu_f = (h * g) * f(e)$$
$$= (h * f) * g(e)$$

$$= \int_{\widehat{G}} \widehat{h} \widehat{f} d\mu_g$$

which implies $\widehat{g} d\mu_f = \widehat{f} d\mu_g$ since

$A(\widehat{G})$ is dense in $C_0(\widehat{G})$. \blacksquare

This Lemma says that formally
 $\frac{d\mu_f}{\widehat{f}}$ is independent of f ; our
strategy is to show that it gives a
Haar measure on \widehat{G} . Of course the
fact that \widehat{f} can have zeros creates a
problem.

In the next lemma we will use the observation that if $f \in L^2(G)$ then $f * f^*$ is well defined and in $L^2(G)$.

Indeed :

$$\begin{aligned} f * f^*(x) &= \int_G f(xy) \overline{f(y)} d\lambda(y) \\ &= \langle \lambda(x)f, f \rangle \end{aligned}$$

where $\lambda: G \rightarrow U(L^2(G))$ is the (continuous) unitary representation of G into $L^2(G)$ defined by

$$\lambda(g)f(y) = f(\bar{g}y).$$

Lemma 9.10 Given any $K \subset \hat{G}$

compact, there is $g \in C_c(G) \cap L^2(G)$ such that $\hat{g} > 0$ on K .

Proof: We know that for every $\gamma \in K$

there is $u_\gamma \in C_0(G)$ with

$$\hat{u}_\gamma(\gamma) \neq 0.$$

Thus there is an open set $V_\gamma \ni \gamma$ with

$$\hat{u}_\gamma(\gamma) \neq 0 \quad \forall \gamma \in V_\gamma.$$

Now $K \subset \bigcup_{\gamma \in K} V_\gamma$; let $\gamma_1, \dots, \gamma_n \in K$

such that $K \subset \bigcup_{i=1}^n V_{\gamma_i}$. Let

$$g := \sum_{i=1}^n u_{\gamma_i} * u_{\gamma_i}^* \in C_0(G) \cap P(G).$$

$$\text{Then } \hat{g}(\gamma) = \sum_{i=1}^n (\hat{u}_{\gamma_i}(\gamma))^2 \geq 0$$

$\forall \gamma \in K$.



Proof of Thm 9.8

Let $\psi \in C_0(\widehat{G})$; pick $K \supset \text{supp}(\psi)$

compact and $g \in C_0(G) \cap P(G)$,

with $\widehat{g} > 0$ on K and define

$$T_{K,g}(\psi) := \int_K \frac{\psi}{\widehat{g}} d\mu_g.$$

If K', g' is another such pair, we

have

$$T_{K,g}(\psi) = \int_{K \cap K'} \frac{\psi}{\widehat{g}} d\mu_g = \int_{K \cap K'} \frac{\psi \widehat{g}'}{\widehat{g}' \cdot \widehat{g}} d\mu_g$$

From Lemma 9.9 we get $\widehat{g}' d\mu_g = \widehat{g} d\mu_g$,

and hence

$$= \int_{K \cap K'} \frac{\psi \widehat{g}}{\widehat{g}' \widehat{g}} d\mu_{g'} = \int_{K \cap K'} \frac{\psi}{\widehat{g}'} d\mu_{g'}$$

$$= \int_{K'} \frac{\psi}{\widehat{g}'} d\mu_{g'} = T_{K',g'}(\psi).$$

Thus $T_{K,j}(\gamma)$ is independent of the choice of K and j such that $K \supset \text{supp } \gamma$ and $\hat{g} > 0$ on K .

We claim that $T: C_0(\hat{G}) \rightarrow \mathbb{C}$ is a positive linear and \hat{G} -invariant functional.

Linearity: clear, since given γ_1, γ_2 in $C_0(G)$ we can always find K compact $K \supset \text{supp } \gamma_1 \cup \text{supp } \gamma_2$ and $g \in C_0(G) \cap P(G)$ with $\hat{g} > 0$ on K .

Positivity: let $\gamma \in C_0(G)$, $\gamma \geq 0$.

Let $K \supset \text{supp } \gamma$ and $g \in C_0(G) \cap P(G)$ with $\hat{g} > 0$ on K . Since $g \in P(G)$ the bounded measure μ_g is positive

and hence $T(\psi) = \int \frac{\psi}{\hat{g}} d\mu_g \geq 0$.

Invariance : Let $\psi \in C_0(\widehat{G})$; choose
 $K \supset \text{supp } \psi$ compact and take
 $g \in C_0(G) \cap P(G)$ with $\widehat{g} > 0$ on
 $K \cup g^{-1}K$.

Let $f(x) := \overline{(x, \delta_0)} g(x)$. Then,

$$f(x) = \int_{\widehat{G}} (x, \delta_0^{-1} s) d\mu_g(s)$$

and hence $f = \tilde{\mu}_f$ where μ_f
is the positive bounded measure defined

$$\text{by } \int \psi(s) d\mu_f(s) := \int \psi(\delta_0^{-1} s) d\mu_g(s)$$

Observing that $\widehat{f}(s) = \widehat{g}(s - \delta_0)$ we

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get:

$$\begin{aligned} T(\lambda \varphi) \psi &= \int \frac{\varphi(\gamma^{-1} \gamma)}{\hat{g}(\gamma)} d\mu_g(\gamma) \\ &= \int \frac{\varphi(\gamma)}{\hat{g}(\gamma \gamma_0)} d\mu_g(\gamma) = \int \frac{\varphi(\gamma)}{\hat{f}(\gamma)} d\mu_g(\gamma) \\ &= T(\psi). \end{aligned}$$

Thus there is exactly one Haar measure
w on \widehat{G} representing T .

Let then $\varphi \in C_c(\widehat{G})$ and $f \in L^1(G) \cap S(G)$.

Pick $g \in C_c(G) \cap r(G)$ with $\hat{g} > 0$

on $K = \text{supp } \varphi$. Then:

$$\begin{aligned} \int \varphi d\mu_g &= \int \frac{\varphi}{\hat{g}} \hat{g} d\mu_g = \int \frac{\varphi}{\hat{g}} \hat{f} d\mu_g \\ &= T(\varphi \hat{f}) = \int \varphi \hat{f} dw \end{aligned}$$

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and hence $\mu_f = \int_G dw$ which proves
the inversion formula. \square

We turn to an important consequence
of the inversion theorem.

Recall from chapter 7 that for
 $C \subset \widehat{G}$ compact and $r > 0$ we
have defined

$$N(C, r) = \left\{ x \in G : |(x, y)^{-1}| < r \right. \\ \left. \forall y \in C \right\}$$

and have shown that it is an open
set; in fact it is an open neighborhood
of $e \in G$. Now we show

Corollary 9.11 $\left\{ \mathbb{E}^{g, N(C, r)} : g \in G, \right.$
 $C \subset G \text{ compact}, r > 0 \left. \right\}$ is a basis
of the topology of G .

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Proof: Let $V \ni e$ be an open neighborhood; let $W \ni e$ be a compact neighborhood of e with $WW^{-1} \subset V$. Define

$f = X_w / \lambda(w)$ and set

$$g = f * f^*$$

The strategy is to find a compact $G \subset \hat{G}$ such that $\forall x \in N(c, \gamma_3)$,

$$g(x) > 0$$

which will imply $x \in WW^{-1} \subset V$

and hence $N(c, \gamma_3) \subset V$.

To this end we apply the inversion theorem to g , which we may since

$$\cancel{g \in L^1(G, \mathbb{R})} \quad g \in C_0(G) \cap P(G).$$

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$$\text{So } g(x) = \int_{\hat{G}} \hat{g}(\gamma)(x, \gamma) d\omega(\gamma). \quad (*)$$

First, evaluate at $x = e$; observe

$$g(e) = 1 \text{ and } \hat{g}(\gamma) = |\hat{f}(\gamma)|^2 > 0 \forall \gamma.$$

Thus $1 = \int_{\hat{G}} \hat{g}(\gamma) d\omega(\gamma)$

and there is thus $\hat{G} \subset \hat{\mathbb{C}}$ compact

with

$$\int_{\hat{G}} \hat{g}(\gamma) d\omega(\gamma) > \frac{2}{3}.$$

Let now $x \in N(G, \gamma_3)$. Write (*)

$$g(x) = \int_G \hat{g}(\gamma)(x, \gamma) d\omega(\gamma) + \int_{\hat{G} \setminus G} \hat{g}(\gamma)(x, \gamma) d\omega(\gamma)$$

Now $\left| \int_{\hat{G} \setminus G} \hat{g}(\gamma)(x, \gamma) d\omega(\gamma) \right| \leq \int_{\hat{G} \setminus G} \hat{g}(\gamma) d\omega(\gamma)$
 $< \frac{1}{3}.$

Also

$$g(x) = \int_{\mathbb{C}} |\hat{g}(z)| \operatorname{Re}(x, z) dw(z) + \operatorname{Re} \left(\int_{G \cap C} \hat{g}(z) dz \right)$$

Observe that if $|1 - (x, z)| < \gamma_3$ then

$\operatorname{Re}(x, z) > \frac{2}{3}$ and hence

$$\int_{\mathbb{C}} |\hat{g}(z)| \operatorname{Re}(x, z) dw(z) > \frac{2}{3} \int_{\mathbb{C}} |\hat{g}(z)| dw(z) \\ > \frac{4}{9}.$$

Thus $g(x) > \frac{4}{9} - \frac{1}{3} = \frac{1}{9}$. □

Corollary 9.12 \hat{G} separates points in G ,
 that is $\forall x_1 \neq x_2$ in G there is
 $x \in \hat{G}$ with $x(x_1) \neq x(x_2)$.

Proof: $\bar{x}_1'x_2 \neq e$; let V be open such that $V \ni e$ and $\bar{x}_1'x_2 \notin V$ and let $G \subset \hat{G}$ be compact such that $N(\zeta, \gamma_3) \subset V$. Then:
 $\bar{x}_1'x_2 \notin N(\zeta, \gamma_3)$ hence there is $\chi \in G$ with $|\chi(\bar{x}_1'x_2)^{-1}| \geq \gamma_3$ which concludes the proof. \square

Examples 9.13

exit

(1) $G = \mathbb{R}$: let $x(t) = e^{it}$

and identify $\hat{\mathbb{R}}$ with \mathbb{R} via

$$\mathbb{R} \rightarrow \hat{\mathbb{R}}$$

$$a \mapsto x_a$$

where $x_a(t) = e^{iat}$. Let \mathcal{L} be
the Lebesgue measure on \mathbb{R} . Then

$$\hat{f}(a) = \int_{\mathbb{R}} f(t) e^{-2\pi i at} d\mathcal{L}(t)$$

and

$$f(t) = \int_{\mathbb{R}} \hat{f}(a) e^{2\pi i at} d\mathcal{L}(a)$$

provided $f \in L^1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$.

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(2) $G = \mathbb{Q}_p$: we fix a continuous character $\chi: \mathbb{Q}_p \rightarrow \overline{\mathbb{U}}$ with $\text{Ker } \chi = \mathbb{Z}_p$, which is possible since the discrete group $\mathbb{Q}_p / \mathbb{Z}_p$ is isomorphic to the subgroup of $\overline{\mathbb{U}}$:

$$\left\{ z \in \mathbb{T} : \exists n \geq 1 \text{ with } z^{p^n} = 1 \right\}.$$

Now identify $\widehat{\mathbb{Q}_p}$ with \mathbb{Q}_p via

$$\mathbb{Q}_p \xrightarrow{\sim} \widehat{\mathbb{Q}_p}$$

$$a \mapsto x_a, \quad x_a|t\rangle = x|at\rangle.$$

Let λ be the Haar measure on \mathbb{Q}_p such that $\lambda(\mathbb{Z}_p) = 1$. One computes then

$$\widehat{\chi}_{\mathbb{Z}_p}(a) = \begin{cases} 1 & \text{if } x_a|_{\mathbb{Z}_p} \equiv 1 \\ 0 & \text{otherwise.} \end{cases}$$

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Now $\chi_a|_{\mathbb{Z}_p} = 1$ if $a \in \mathbb{Z}_p$

$\forall t \in \mathbb{Z}_p$ iff $t \in \mathbb{Z}_p$. Thus:

$$\widehat{\chi}_{\mathbb{Z}_p} = \chi_{\mathbb{Z}_p}.$$

Thus λ is also the right normalized Haar measure for the inverse formula to hold.

(3) If G is compact we have seen that \widehat{G} is discrete. Let λ be the Haar measure on G s.t. $\lambda(G) = 1$.

We have seen that then

$$\widehat{\mathbf{1}}_G(x) = \begin{cases} 1 & x = \delta_{\hat{e}} \\ 0 & x \neq \hat{e} \end{cases}$$

and hence $\widehat{\mathbf{1}}_G(e) = \sum_{x \in \widehat{G}} \delta_{\hat{e}}(x)$

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which implies that the dual Haar measure μ on \widehat{G} is counting measure.