

## 6. Locally compact groups, basic properties and examples.

In this chapter we introduce topological groups, treat some examples with an emphasis on abelian ones; in particular we discuss in some details the field of  $p$ -adics. Then we shortly discuss how topology and group theory interact to produce some miraculous facts about topological groups.

The second part of the chapter is devoted to introducing the Haar measure on a locally compact group and establishing some basic properties of the convolution product.

## 6.1 Topological groups : definitions and first examples.

Let  $G$  be a group.

Def 6.1. A topology  $\mathcal{T} \subset \mathcal{P}(G)$  on the set  $G$  endows  $G$  with the structure of topological group if the multiplication map  $G \times G \rightarrow G$ ,  $(x, y) \mapsto x \cdot y$ , and the inverse map  $G \rightarrow G$ ,  $g \mapsto g^{-1}$  are continuous.

In the above definition,  $G \times G$  is endowed with the product topology.

Let's draw some interesting consequences from Def. 6.1 :

Remark 6.2. Let  $G$  be a topological group.

(1) The inverse map  $i: G \rightarrow G, g \mapsto g^{-1}$  is continuous, in addition  $i \circ i = id_G$  and hence  $i$  is a homeomorphism.

(2) For  $g \in G$ , define the left translation by  $g$ :  $L_g: G \rightarrow G, x \mapsto g \cdot x$

which is continuous by definition.

Observe that:  $L_{g^{-1}} \circ L_g = L_g \circ L_{g^{-1}} = id_G$

and hence  $L_g$  is a homeomorphism.

Hence a topological group looks locally everywhere the same.

Analogously one defines right translation by

$R_g: G \rightarrow G, x \mapsto x \cdot g$  and concludes that it is a homeomorphism.

(3) Let  $\varphi: G_1 \rightarrow G_2$  be a homomorphism  
where  $G_1, G_2$  are topological groups.

Assume  $\varphi$  continuous at  $e \in G_1$ .

Then given  $g \in G$  arbitrary,  $\varphi \circ L_{g^{-1}}$   
is continuous at  $g$  since  $L_{g^{-1}}(g) = e$ .

But  $(\varphi \circ L_{g^{-1}})(h) = \varphi(g^{-1}) \cdot \varphi(h)$ ,  
which can also be written as

$$L_{\varphi(g)} \cdot \varphi \circ L_{g^{-1}} = \varphi$$

implying that  $\varphi$  is continuous at  $g \in G$ .

Since  $g$  is arbitrary we have established,  
a homomorphism  $\varphi: G_1 \rightarrow G_2$  is continuous  
 $\iff$  it is continuous at  $e \in G_1$ .

(4) Let  $H \subset G$  be a subgroup of  $G$  then with the induced topology,  $H$  is a topological group.

Now we turn to examples:

Examples 6.3 :

(1) Any group  $G$  endowed with the discrete topology.

(2)  $(\mathbb{R}^n, +)$  equipped with the Euclidean topology is an abelian topological group.

(3) If  $A$  is a unital Banach algebra, the group  $G(A)$  of invertible elements with the topology induced from  $A$  is a topological group (Lemma 2.7).

(4) The additive groups  $(\mathbb{R}, +)$ ,  $(\mathbb{C}, +)$  as well as the multiplicative groups  $(\mathbb{R}^*, \cdot)$ ,  $(\mathbb{C}^*, \cdot)$  of the fields  $\mathbb{R}$ ,  $\mathbb{C}$  are abelian topological groups.

Observe that Examples (1), (2), (4) are locally compact Hausdorff, while  $G(A)$  is locally compact iff  $A$  is finite dimensional (Exercise).

In this context the following examples are a special case of (3)

(5)  $GL(n, \mathbb{R})$  and  $GL(n, \mathbb{C})$  are locally compact Hausdorff groups.

The cartesian product leads to a

wealth of examples:

(c) Let  $G_\alpha$ ,  $\alpha \in A$  be a family of topological groups. Endow  $G = \prod_{\alpha \in A} G_\alpha$  with

the componentwise product and the product topology. Then  $G$  is a topological group. It is compact if  $G_\alpha$  is compact  $\forall \alpha \in A$ . For instance if we endow

$\mathbb{Z}/_{2^k}$  with the discrete topology then

$(\mathbb{Z}/_{2^k})^\text{ur}$  is a compact group. It is

an instructive exercise to show that

$G = \prod_{\alpha \in A} G_\alpha$  is locally compact Hausdorff

iff all  $G_\alpha$  are locally compact Hausdorff  
AND the  $G_\alpha$ 's are compact except  
finitely many.

There is a general source of locally compact groups namely:

(7) Let  $(X, d)$  be a metric space such that all closed balls of finite radius are compact, then the group  $Is(X)$  of isometries of  $(X, d)$  with compact open topology is locally compact Hausdorff.

(8) The ring  $\mathbb{Z}_p$  of  $p$ -adic integers.

For every  $n \geq 1$ , let  $A_n := \mathbb{Z}_{p^n \mathbb{Z}}$ , the ring of integers modulo  $p^n$ ; given  $x \in A_n$  its reduction mod  $p^{n-1}$  is well defined and leads to a surjective ring homomorphism

$$\phi_n : A_n \rightarrow A_{n-1}.$$

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We obtain a sequence of rings with

morphisms connecting them:

$$A_1 \xleftarrow{\phi_2} A_2 \xleftarrow{\phi_3} A_3 \cdots \xleftarrow{\phi_n} A_n \leftarrow A_{n+1}$$

The ring  $\mathbb{Z}_p$  of  $p$ -adic integers is the projective limit of the system  $(A_n, \phi_n)$  defined above. By definition  $\mathbb{Z}_p$  is the subring of  $\prod_{n \geq 1} A_n$  given by:

$$\mathbb{Z}_p = \left\{ (x_1, x_2, x_3, \dots) \in \prod_{n \geq 1} A_n : \right.$$

$$\left. \phi_n(x_n) = x_{n-1}, \quad \forall n \geq 1 \right\}.$$

More precisely, coordinate wise addition and multiplication on  $\prod_{n \geq 1} A_n$  makes it a ring; since the  $\phi_n$  are ring homomorphisms,  $\mathbb{Z}_p$  is then a subring.

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Bottom

The ring  $\mathbb{Z}$  injects into  $\mathbb{Z}_p$  via

$$\mathbb{Z} \rightarrow \mathbb{Z}_p$$

$$x \mapsto (x \pmod p, x \pmod{p^2}, \dots)$$

and we identify it with a subring of  $\mathbb{Z}_p$ .

Now endow  $A_n$  with discrete topology,

then  $\prod_{n \geq 1} A_n$  is compact Hausd.

$\mathbb{Z}_p$  being defined by closed conditions

is hence compact Hausdorff. Both

operations of addition and multiplication

are continuous and so is  $x \mapsto -x$ .

In particular  $(\mathbb{Z}_p, +)$  is an abelian

compact Hausdorff group. Observe that

$\mathbb{Z}$  is dense in  $\mathbb{Z}_p$ .

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Essential Exercise It is well known  
that if  $p \equiv 1 \pmod{4}$   $\rightarrow$  is a square in  
 $\mathbb{Z}/p\mathbb{Z}$ , that is,  $x^2 + 1 \equiv 0$  has a  
solution in  $\mathbb{Z}/p\mathbb{Z}$ . Use this and elementary  
computations to ~~show that we~~ inductively  
construct a sequence  $x_n \in \mathbb{Z}/p^n\mathbb{Z}$  with

$$(1) \quad x_n^2 + 1 \equiv 0 \text{ in } \mathbb{Z}/p^n\mathbb{Z}$$

$$(2) \quad \phi(x_n) = x_{n-1}.$$

Thus  $x^2 + 1 \equiv 0$  has a solution in  $\mathbb{Z}_p$ .

Important in the study of the ring  $\mathbb{Z}_p$   
is its relation to the  $A_n$ 's. In fact  
associating to  $x = (x_1, x_2, \dots) \in \mathbb{Z}_p$  its  
 $n$ 'th component  $x_n \in A_n$  defines a  
continuous ring hom.  $\varepsilon_n : \mathbb{Z}_p \rightarrow A_n$

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whose kernel is  $p^n \mathbb{Z}_p$ . The latter is hence an open subgroup of  $\mathbb{Z}_p$  and since  $\bigcap_{n \geq 1} p^n \mathbb{Z}_p = (0)$ , they form a fundamental system of neighborhoods of  $(0)$ .

We have

- (1)  $x \in \mathbb{Z}_p$  is invertible  $\Leftrightarrow x \notin p \mathbb{Z}_p$ .
- (2) if  $U \subset \mathbb{Z}_p$  denotes the group of invertible elements, then every  $x \in \mathbb{Z}_p \setminus U$  can be written uniquely as  $x = p^n \cdot u$  with  $n \geq 0$  and  $U \ni u$ .

The first property follows from the analogous statement for  $A_n$  and the second from  $\bigcap_{n \geq 1} p^n \mathbb{Z}_p = (0)$ .

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(g) The field  $\mathbb{Q}_p$ : observe that  $\mathbb{Z}_p$  is an integral domain. Indeed if  $x \cdot y = 0$  and  $x \neq 0, y \neq 0$  write  $x = p^n u$ ,  $y = p^m u'$  with  $u, u' \in U$ ; then  $x \cdot y = p^{n+m} u u' = 0$  hence  $p^{n+m} = 0$  which is nonsense. One deduces easily that

$\mathbb{Q}_p$  can be identified with  $\mathbb{Z}_p[\bar{p}^{-1}]$ .

In fact every  $x \in \mathbb{Q}_p^\times$  can be written uniquely as  $x = p^n u$ ,  $n \in \mathbb{Z}, u \in U$

If we set  $v(x) := n \in \mathbb{Z}$  we obtain

a so-called valuation, that is

$$v : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{-\infty\}$$

where we put  $v(0) = -\infty$ , satisfying

$$v(xy) = v(x) + v(y)$$

$$v(x+y) \geq \min(v(x), v(y)).$$

It follows that  $d(x, y) := e^{-\varphi(x-y)}$

defines a distance on  $\mathbb{Q}_p$ : it induces  
the given topology on  $\mathbb{Z}_p$ . One concludes:

$\mathbb{Q}_p$  is locally compact and contains  
 $\mathbb{Z}_p$  as an open subring; the field  
 $\mathbb{Q}$  is dense in  $\mathbb{Q}_p$ .

For (slightly) more details see Serre,  
"A course in Arithmetic" Chapter II, §1.

The following illustrates the role that  
the above fields play in a local approach  
to algebraic number theory. See A.-Weil

"Basic Number Theory".

(1) Assume  $K$  is a non-discrete, locally  
compact Hausdorff field of  $\text{char } K = 0$ .

Then  $K$  is isomorphic to  $\mathbb{R}$ ,  $\mathbb{C}$ , or  
a finite extension of  $\mathbb{Q}_p$  for some  $p$ .

(2) If  $i: \mathbb{Q} \hookrightarrow K$  is an injection  
with dense image and  $K$  is loc. compact  
non-discrete, then  $K = \mathbb{R}$  or  $\mathbb{Q}_p$ , for  
some  $p$ .

## 6.2. Some miraculous facts about topological groups.

Recall that a topological space is connected if it cannot be written as disjoint union of open non-empty subsets. Recall that the closure of a connected set is connected and the continuous image of a connected set is connected. Finally, given a topological space  $X$ , the relation  $x \sim y$  if  $\{x, y\}$  is contained in a connected subset of  $X$  is an equivalence relation and its equivalence classes are called connected components.

Prop. 6.4. Let  $G$  be a topological group

Then we have

(1) If  $H \leq G$  is a subgroup, so is  
its closure  $\overline{H}$ .

(2) If  $H < G$  is an open subgroup then  
it is closed.

(3) The connected component  $G_0$  of  $G$   
containing the identity is a closed  
normal subgroup.

(4) If  $G$  is connected and  $V \ni e$   
is a neighborhood of  $e$  then  $V \cup \bar{V}'$   
generates  $G$ , that is:

$$G = \bigcup_{n \geq 1} (V \cup \bar{V}')^n.$$

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Notation give subset  $A, B \subset G$  we denote  $A \cdot B := \{a \cdot b \mid a \in A, b \in B\}$

$$\bar{A}^{\circ} = \{ \bar{a} : a \in A \}$$

$$A^n = A \cdot \bar{A}^{n-1} \quad n \geq 2.$$

Proof:

(1) Recall that given  $f: X \rightarrow F$

continuous,  $A \subset X$  then  $f(\bar{A}) \subset \overline{f(A)}$ .

Applying this to the multiplication map

$$m: G \times G \rightarrow G  
(x, y) \mapsto xy$$

and the inversion

$$i: G \rightarrow G  
x \mapsto x^{-1}$$

we get:

$$m(\bar{H} \times \bar{H}) = m(\overline{H \times H}) \subset \overline{m(H \times H)}  
= \overline{H}.$$

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And  $i(\bar{H}) \subset \overline{i(H)} = \bar{H}$

which shows (1).

(2) Let  $R \subset G$  be a complete set of representatives for  $G/H$ , that is:

$$G = \bigsqcup_{x \in R} x \cdot H = H \sqcup \bigsqcup_{x \in R \setminus \{e\}} L_x(H)$$

where  $\bigsqcup$  denotes disjoint union.

Since  $H$  is open and  $L_x : G \rightarrow G$  a

homom. so is  $L_x(H)$  and hence

$\bigsqcup_{x \in R \setminus \{e\}} L_x(H)$  which implies that  $H$  is closed.

(3) Notice first that connected components are always closed. Thus the subset  $G_0$

is closed. Now  $m(G_0 \times G_0) \subset G$

and  $i(G_0) \subset G$  are connected subset containing  $e \in G$ . Hence they are contained

in  $G_0$  which shows that  $G_0$  is a subgroup. Finally given  $g \in G$ , and observing that  $G \rightarrow G, x \mapsto gx\bar{g}'$  is continuous we have that

$$\{gx\bar{g}': x \in G_0\}$$

is a connected subset of  $\overset{\circ}{G}$  containing  $e$  hence is contained in  $G_0$ . This shows that  $G_0$  is normal in  $G$ .

(4) Observe that  $H := \bigcup_{n \geq 1} (V \cup V')$ <sup>n</sup> is a subgroup of  $G$ . Now let  $u \in U \subset V$  be an open subset of  $G$ ; then  $U \subset H$  and since  $H$  is a group we have

$$L_h(u) \subset H \quad \forall h \in H.$$

But  $L_h(u) \ni h$  is an open neighborhood of  $h$ ; hence  $H$  is a neighborhood of

each of its points hence  $H$  is open.

By (2)  $H$  is hence closed; since  $G$  is connected one  $H \neq \emptyset$  we deduce that  $H = G$ .



Remark 6.5 According to (3) we can write  $G = \bigsqcup x G_0$  where the union is over a complete set of representatives of  $G/G_0$ . Hence the set  $\pi_0(G)$  of connected components of  $G$  acquires via its identification with  $G/G_0$  a group structure. It is an instructive exercise to compute  $\pi_0(G)$  in each of the examples 6.2.

### 6.3. Haar measure and convolution product.

Let now  $G$  be locally compact Hausdorff. The (arguably) most important fact about this class of groups is the existence and uniqueness of Haar measure which we now describe in more detail.

Let as usual  $C_0(G)$  be the  $\mathbb{C}$ -vector space of continuous compactly supported functions on  $G$ . Given any map  $F: G \rightarrow X$  into a set  $X$ , we denote  $\lambda(g)F: G \rightarrow X$  the map  $(\lambda(g)F)(x) = \widetilde{f}(g^{-1}x)$ .

Clearly if  $f \in C_c(G)$  then  $\lambda(g)f \in C_c(G)$   
and one verifies easily that this way  
one obtains a group homomorphism

$$\lambda: G \rightarrow GL(C_c(G)).$$

The fundamental theorem is then:

Theorem 6.6. ~~Let~~ Let  $G$  be a locally  
compact Hausdorff group. Then there  
exists a non-zero, positive linear  
functional  $\lambda: C_c(G) \rightarrow \mathbb{C}$   
that is invariant under left translation,

i.e.  $\lambda(\lambda(g)f) = \lambda(f), \forall g \in G$   
 $\# f \in C_c(G).$

Moreover given two such functionals  
 $\lambda_1, \lambda_2$  there exists  $c > 0$  with

$$\lambda_2 = c \cdot \lambda_1.$$

Such a functional is called a ~~#~~ left Haar-funct.

Using Riesz' representation theorem  
one obtains an equivalent formulation:

Corollary 6.7 There is, up to strictly positive  
scalar multiple, a unique non-zero, positive  
regular Borel measure  $\mu$  on  $G$  such that  
for every measurable set  $E \subset G$  and  
every  $g \in G$ :  $\mu(gE) = \mu(E)$ .

For the proof of Haar's theorem we refer  
e.g. to A. Weil, "L'intégration dans les groupes  
topologiques et ses applications".

A measure as in Corollary 6.7 is called  
a left Haar measure. In case  $G$  is  
abelian we have  $L_g = R_g \ \forall g \in G$  and  
we call a left Haar measure just Haar measure.

## Examples 6.8

(1) The Lebesgue measure  $\mathcal{L}$  on  $\mathbb{R}$ ,

that is, the unique positive regular Borel

measure on  $\mathbb{R}$  such that  $\mathcal{L}([a, b]) = b - a$

for  $a \leq b$  is a Haar measure for  $(\mathbb{R}, +)$

and so is  $\underbrace{\mathcal{L} \times \dots \times \mathcal{L}}_{n \text{ times}}$  for  $(\mathbb{R}^n, +)$ .

(2) A Haar measure for the multiplication group  $(\mathbb{R}^\times, \cdot)$  is given by

$$d\mu(x) = \frac{d\mathcal{L}(x)}{|x|}.$$

(3) Let  $G$  be discrete, then

$$\mu(E) = \text{card}(E), \quad \forall E \subseteq G \text{ is}$$

a left Haar measure.

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The uniqueness statement is probably more important than the existence statement, since it gives us immediately additional structure.

Corollary 6.9 Let  $\text{Aut}(G)$  denote  
the group of continuous automorphisms  
of  $G$ . Then there is a well-defined  
group homomorphism

$$\text{mod}_G : \text{Aut}(G) \rightarrow \mathbb{R}_{>0}$$

into the multiplicative group  $(\mathbb{R}_{>0}, \cdot)$   
such that for any left Haar functional

$$\Lambda : \Lambda(f \circ \alpha^{-1}) = \text{mod}_G(\alpha) \Lambda(f)$$

$$\forall f \in C_0(G)$$

$$\forall \alpha \in \text{Aut } G.$$

Proof.: Observe that  $f \mapsto f \circ \bar{\alpha}'$  is

a linear map on  $C_0(G)$  preserving positivity

In addition we have for  $\alpha \in \text{Aut } G$ ,

$f \in C_0(G)$  and  $g, x \in G$ :

$$(\lambda(g)f)(\bar{\alpha}(x)) = f(g^{-1}\bar{\alpha}(x)) = f(\bar{\alpha}^{-1}(g^{-1}\bar{\alpha}(x))) \\ = \lambda(\bar{\alpha}(g))(f \circ \bar{\alpha})(x)$$

And as a result if we define

$$\Lambda_\alpha^*(f) := \lambda(f \circ \bar{\alpha})$$

then  $\Lambda_\alpha^*$  is a non-zero positive functional

$$\text{and } \Lambda_\alpha^*(\lambda(g)f) = \lambda((\lambda(g)f) \circ \bar{\alpha})$$

$$= \lambda(\lambda(\bar{\alpha}(g))(f \circ \bar{\alpha})) = \lambda(f \circ \bar{\alpha}) = \Lambda_\alpha^*(f)$$

and hence  $\Lambda_\alpha^*$  is a Haar function!

By uniqueness there is a constant

$$c_\Lambda > 0 \text{ with } \Lambda_\alpha^* = c_\Lambda \Lambda.$$

One verifies easily that  $c_\lambda(\alpha)$  is independent on the choice of  $\lambda$ , and defines

$$\text{mod}_C(\alpha) = c_1(\alpha).$$

Then from  $\lambda_{\alpha_1, \alpha_2} = (\lambda_{\alpha_1})_{\alpha_2}$

$$= c_{\lambda_{\alpha_1}}(\alpha_2) \lambda_{\alpha_1} = c_{\lambda_{\alpha_1}}(\alpha_2) c(\alpha_1) \wedge$$

and the independence of  $c_\lambda$  on  $\lambda$

follows:  $\text{mod}_C(\alpha_1 \alpha_2) = \text{mod}_C(\alpha_1) \text{mod}_C(\alpha_2).$

□

### Example 6.10.

(1) Let  $G = (\mathbb{R}^n, +)$  then (exercise)

$\text{Aut } G = GL(n, \mathbb{R})$  and:

$$\text{mod}_G(\alpha) = \frac{\det \alpha}{|\det \alpha|} \quad (\det \alpha)$$

Indeed if  $\mathcal{L}$  is Lebesgue measure on  $\mathbb{R}^n$  and  $\alpha \in GL(n, \mathbb{R})$  then

$$\mathcal{L}(\alpha([0,1]^n)) = |\det \alpha| \cdot \mathcal{L}([0,1]^n).$$

This translates into the statement

$$\int_{[0,1]^n} X(\alpha^{-1}(x)) d\mathcal{L}(x) = |\det \alpha| \int_{[0,1]^n} X(x) d\mathcal{L}(x)$$

but the left hand side equals

$$\text{mod}_G(\alpha^{-1}) \int_{[0,1]^n} X(x) d\mathcal{L}(x).$$

(2) Let  $\mathbb{K}$  be a locally compact field.

One gets this way two locally compact (abelian) groups, namely  $(\mathbb{K}, +)$  and  $(\mathbb{K}^\times, \cdot)$  which both have Haar measures. Now with the identification

$$\mathbb{K}^\times = GL(1, \mathbb{K}) \hookrightarrow \text{Aut}(\mathbb{K}, +)$$

We obtain a canonical homomorphism

$$\text{mod}_{\mathbb{K}} : \mathbb{K}^\times \longrightarrow R_{>0}$$

which if  $\mu$  is any Haar measure

on  $(k, +)$  verifying the Heck-Borel substa-

$$\mu(y \cdot E) = \text{mod}_k(y) \mu(E)$$

$$\forall y \in k^*.$$

A non-trivial fact is that if  $k$  is  
non-discrete then  $y \mapsto \text{mod}_k(y)$

is not identically  $= 1$  and behavior  
like an absolute value, that is  $\exists c > 0$

$$\text{such that } \text{mod}_k(y_1 y_2) \leq c \cdot \max(\text{mod}_k(y_1), \text{mod}_k(y_2))$$

This is the starting point of the classification  
of non-discrete locally compact fields,

which can be found in A. Weil,

"Basic Number Theory" Chap I.

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(3) It is not so easy to write down a Haar measure for  $(\mathbb{Q}_p, +)$ . However it is not difficult to determine  $\text{mod}(y)$  for  $y \in \mathbb{Q}_p$ . Indeed

We may assume  $y \in \mathbb{Z}_p$ , since  $\text{mod}(y')$   $= \text{mod}(y)$ . Write  $y = p^n \cdot u$ ,  $n \geq 0$

and  $u \in U$  invertible in  $\mathbb{Z}_p$ . Now

$\mathbb{Z}_p$  is open in  $\mathbb{Q}_p$  and compact so if  $\mu$  is a Haar measure on  $\mathbb{Q}_p$ ,

$\mu(\mathbb{Z}_p) > 0$ . On one hand we have  $\mu(y \cdot \mathbb{Z}_p) = \text{mod}(y) \mu(\mathbb{Z}_p)$

on the other:  $y \mathbb{Z}_p = p^n \mathbb{Z}_p$  and the latter is the kernel of the surjective hom.  $\Sigma_n : \mathbb{Z}_p \rightarrow A_n = \mathbb{Z}/p^n \mathbb{Z}$ .

Let then  $R$  be a complete set of  
representatives of  $\mathbb{Z}_p / p^n \mathbb{Z}_p$ ; so

$$|R| = p^n \text{ and } \mathbb{Z}_p = \bigsqcup_{r \in R} (r + p^n \mathbb{Z}_p)$$

which implies

$$\begin{aligned} \mu(\mathbb{Z}_p) &= \sum \mu(r + p^n \mathbb{Z}_p) \\ &= p^n \cdot \mu(p^n \mathbb{Z}_p) \end{aligned}$$

Hence  $\mu(p^n \mathbb{Z}_p) = p^{-n} \mu(\mathbb{Z}_p)$

and hence  $\text{mod}(y) = p^{-n}$ .