

The following theorem is one of the most fundamental results in the theory of Banach algebras. It uses some basic facts from the theory of holomorphic functions. It also uses the uniform boundedness theorem from the theory of Banach spaces.

Theorem 2.8 Let A be a Banach algebra and $x \in A$. Then the spectrum $\text{Sp}_A(x)$ is a non-empty compact subset of \mathbb{C} and:

$$r_A(x) = \max \{ |\lambda| : \lambda \in \text{Sp}_A(x) \}.$$

then $\lambda \notin \text{Sp}_A(x)$ which implies

$$\|x\| = \sup_{\text{Sp}} \{ |\lambda| : \lambda \in \text{Sp}_A(x) \} \leq r_A(x).$$

Observe also at this point that $\text{Sp}_A^{(sc)}$ is a closed subset of \mathbb{C} ; indeed

the map $\mathbb{C} \rightarrow A$ is continuous
 $\lambda \mapsto x - \lambda e$

and $\text{Sp}_A(x)$ is the inverse image of $A \setminus G(A)$ which is closed by Lemma 2.7(2).

Now we show that $\text{Sp}_A(x) \neq \emptyset$:

This will involve Liouville's theorem.

To this end fix any $\ell \in A^*$, that is, continuous linear form $\ell: A \rightarrow \mathbb{C}$.

and define $\forall \lambda \in P_A(x) = \mathbb{C} \setminus \mathbb{S}_{P_A}(x)$

$$f(\lambda) = \ell((\lambda e^{-x})^{-1}).$$

For μ, λ in $P_A(x)$ we compute:

$$f(\lambda) - f(\mu) = \ell((\lambda e^{-x})^{-1} - (\mu e^{-x})^{-1}).$$

Since $(\lambda e^{-x}), (\mu e^{-x})$ commute in the group $G(A)$ we get:

$$= \ell\left(\frac{(\mu e^{-x}) - (\lambda e^{-x})}{(\lambda e^{-x})(\mu e^{-x})}\right)$$

$$= (\mu - \lambda) \ell\left(\frac{1}{(\lambda e^{-x})(\mu e^{-x})}\right)$$

Thus if $\mu \neq \lambda$:

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -\ell\left(\frac{1}{(\lambda e^{-x})(\mu e^{-x})}\right)$$

Now the map $y \mapsto y^{-1}$ from $\mathcal{C}(A) \rightarrow \mathcal{G}(A)$ basing contours we conclude

$$\lim_{\mu \rightarrow \lambda} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -\ell \left(\frac{1}{(\lambda - x)^2} \right)$$

and hence $f: P_A(x) \rightarrow \mathcal{C}$ is a holomorphic function on the open subset $P_A(x)$.

Next we will derive a power series expansion of f in $\frac{1}{\lambda}$ for large λ ; this will allow us to establish the behaviour of f at ∞ and eventually to establish the spectral radius formula.

We know that if $|\lambda| > r_A(x)$

then $r_A\left(\frac{x}{\lambda}\right) < 1$ and hence by Lemma 2.6

we have the convergent series expansion:

$$\left(e - \frac{x}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{x}{\lambda}\right)^n$$

that is

$$\left(\lambda e - x\right)^{-1} = \sum_{n=-\infty}^{\infty} \lambda^{-(n+1)} x^n.$$

Thus

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^{-(n+1)} \ell(x^n).$$

which converges for $|\lambda| > r_A(x)$.

Now we show that $\text{Sp}_A(x) \neq \emptyset$.

By contradiction: assume $\text{Sp}_A(x) = \emptyset$

then f is holomorphic on \mathbb{C} .

Let now $|\lambda| \geq 2\|x\|$, we estimate

by :

$$f(\lambda) = \lambda' \sum_{n=0}^{\infty} \lambda^{-n} e(x^n)$$

here.

$$\|f(x)\| \leq \|x\|^{\alpha} \sum_{n=0}^{\infty} \left(2\|x\|\right)^{-n} \|e^t\| \|x\|^{\alpha}$$

which implies $\lim_{|x| \rightarrow \infty} |f(x)| = 0$. (*)

In particular f is bounded; being holomorphic ~~it is~~ on \mathfrak{I} it is hence constant (Liouville) and this constant has to vanish by (*).

Thus we conclude that if $\text{Sp}_A(x) = \emptyset$
 then $\ell((x_e - x_i)^*) = 0 \quad \forall \ell \in A^*$

which implies $(\lambda c - x)^{-1} = 0$, a contradiction.

→ 2-~~19~~ -

Finally we proceed to show that

$$r_A(x) \leq \|x\|_{sp}.$$

Define for say $0 < |\beta| < \frac{1}{r_A(x)}$

$$\begin{aligned} g(\beta) &:= f(\beta x) \\ &= \sum_{n=0}^{\infty} \beta^{n+1} l(x^n). \quad (***) \end{aligned}$$

Which is convergent. Then g extends to a holomorphic function at $\beta = 0$.

Moreover since f is holomorphic in

$$\mathbb{C} \setminus \text{Sp}_A(x) \supset \mathbb{C} \setminus \{ \lambda \in \mathbb{C} : |\lambda| \leq \|x\|_{sp} \}$$

it follows that g is holomorphic in

$$\left\{ \beta \in \mathbb{C} : |\beta| < \frac{1}{\|x\|_{sp}} \right\}$$

and hence by a fundamental theorem
in the theory of holomorphic functions
the Taylor expansion $(**)$ of g at 0
converges $\forall |z| < \|x\|_{sp}^{-1}$; in
particular:

$$\sup_{n \geq 1} |\ell(z^n \cdot x^n)| < +\infty$$

$$\forall \ell \in A^*$$

which by the uniform boundedness principle
implies that

$$\sup_{n \geq 1} \|z^n \cdot x^n\| := C(z) < +\infty$$

$$\forall |z| < \|x\|_{sp}^{-1}.$$

From the above inequality we get

$$\|x^n\|^{\gamma_n} \leq C(z)^{\gamma_n} |z|^{-1}$$

hence $A(x) \leq |z|^{-1}$, $\forall |z| < \|x\|_{sp}^{-1}$

which implies $\|f_A(x)\| \leq \|x\|_{S^p}$ and concludes the proof of the theorem. \square

The following corollary is the Guelfand-Mazur theorem that is basic to much of Guelfand's theory discussed in the next section and generalizes Frobenius theorem which says that a finite dimensional \mathbb{C} algebra in which every non-zero element is invertible is \mathbb{C} itself.

Corollary 2.9. Let A be a Banach algebra with identity e and suppose that every element in $A \setminus \{0\}$ is invertible. Then A is canonically isomorphic to \mathbb{C} .

Proof: Let $x \in A$: since $\text{Sp}_A(x) \neq \emptyset$

there is $\lambda \in \mathbb{C}$ such that $x - \lambda e$ is not invertible; but then $x - \lambda e = 0$ and

λ is uniquely determined by x , call it $\lambda(x)$. Then $A \rightarrow \mathbb{C}$ is the desired

$$x \mapsto \lambda(x)$$

\mathbb{C} -algebra isomorphism. This is canonical because \mathbb{C} does not admit any other

\mathbb{C} -algebra automorphisms than $\text{Id}_{\mathbb{C}}$



We indicate here natural developments which are part of a more comprehensive treatment of the theory of Banach algebras. This is the holomorphic functional calculus; we will however not use those results (ver m.).

Let A be a unital Banach algebra. For every polynomial

$$P(x) = a_n x^n + \dots + a_0 \in \mathbb{C}[x]$$

and element $x \in A$,

$$P(x) := a_n x^n + \dots + a_1 x + a_0 \cdot 1 \in A$$

is well defined. In fact, this can be extended to a ring of holomorphic functions in the following way:

Let $x \in A$, $\text{Sp}_A(x) \subset \mathbb{C}$ and
 $f: U_f \rightarrow \mathbb{C}$ holomorphic with Σ_f
open containing $\text{Sp}_A(x)$. Let $C \subset U_f$
be a simple closed curve enclosing



$\text{Sp}_A(x)$. For every $\gamma \in A^*$ define

$$F(\gamma) := \frac{1}{2\pi i} \int_C f(z) \gamma((z-x)^{-1}) dz$$

Prop. 2.10. $F \in A^{**}$ is in fact represented
by a vector $v \in A$, that is,

$$F(\gamma) = \gamma(v) \quad \forall \gamma \in A^*.$$

We will denote this vector by $f(x) \in A$;
symbolically $f(x)$ is given by :

$$f(x) = \frac{1}{2\pi i} \int_C f(\lambda) (\lambda - x)^{-1} d\lambda.$$

Prop. 2.11: The map $f \mapsto f(x) \in A$ is
a homomorphism of the Algebra
 $A(Sp_A(x))$ of all functions holomorphic
in a neighbourhood of $Sp_A(x)$, into A
which sends the constant function to the
identity $1 \in A$ and the function
 $\lambda \mapsto \lambda - x$.

Theorem 2.12 : (Spectral mapping Thm.)

Let $x \in A$, unital Banach algebra and
 f holomorphic on a neighborhood of
 $\text{Sp}_A(x)$. Then:

$$\text{Sp}_A(f(x)) = f(\text{Sp}_A(x)).$$

Moreover if g is holomorphic on a
neighborhood of $f(\text{Sp}_A(x))$ then:

$$(g \circ f)(x) = g(f(x)).$$

For all this we refer to Takesaki;
p 8, 10, 11.