

5.4. Resolutions of the identity.

Let X be a compact Hausdorff space,
 \mathcal{B} the σ -algebra of Borel sets and
 \mathcal{L} a Hilbert space.

Definition 5.12 : A resolution of the identity is a map $E : \mathcal{B} \rightarrow \mathcal{L}(H)$ with the following properties:

- (1) $E(\emptyset) = 0$, $E(X) = Id$
- (2) $\forall \omega \in \mathcal{B}$, $E(\omega)$ is a self-adjoint projection.

$$(3) E(\omega_1 \cap \omega_2) = E(\omega_1) \cdot E(\omega_2)$$

$\forall \omega_1, \omega_2$ in \mathcal{B}

(4) If $\omega_1 \cap \omega_2 = \emptyset$ then

$$E(\omega_1 \cup \omega_2) = E(\omega_1) + E(\omega_2)$$

(5) $\forall x \in \mathcal{X}$ the set function

$$E_{x,x}(w) := \langle E(w)x, x \rangle$$

is a positive regular Borel measure
on X .

Here we discuss these conditions and
their consequences.

From (3) follows that $\{E(w) : w \in \mathcal{B}\}$
is a family of commuting self-adjoint
projections.

Also if $\omega_1 \cap \omega_2 = \emptyset$ then

$$0 = E(\emptyset) = E(\omega_1) E(\omega_2)$$

and hence (Prop 5.10) $\text{Im } E(\omega_1) \perp \text{Im } E(\omega_2)$.

Let's consider (4) : since $E(\omega)$ is a self adjoint projection we have

$$E_{x,x}(\omega) = \langle E(\omega)x, x \rangle = \|E(\omega)x\|^2 \geq 0$$

and if $\omega_1 \cup \omega_2 = \emptyset$ then it follows from (4) that

$$E_{x,x}(\omega_1 \cup \omega_2) = E_{x,x}(\omega_1) + E_{x,x}(\omega_2).$$

Thus conditions (1)-(4) imply that $\omega \mapsto E_{x,x}(\omega)$ is a positive additive set function.

Then (5) essentially postulates that this set function is σ -additive.

Let us now define for $x, y \in \mathcal{H}$:

$$E_{x,y}(\omega) := \langle E(\omega)x, y \rangle.$$

Then Prop. 5.7 implies:

$$2 E_{x,y}(\omega) = E_{x+y, x+y}(\omega) + i E_{x+iy, x+iy}(\omega) \\ - (1+i) E_{x,x}(\omega) - (1+i) E_{y,y}(\omega)$$

and since $E_{x,y}$ is a complex measure in the sense of Def. 5.3, and has the σ -additivity property stated in Essential Fact 5.4. Using this we will now show

Prop. 5.13 For every $x \in \mathcal{H}$, $\omega \mapsto E(\omega)x$

is countably additive, that is if

$\omega = \bigsqcup_{n \geq 1} \omega_n$ is a countable disjoint

union of Borel sets then

$$E(\omega)x = \sum_{n \geq 1} E(\omega_n).x$$

where the series converges in the norm topology of \mathcal{H} .

We will need :

Lemma 5.14 : Assume $\{x_n : n \geq 1\}$

is a sequence of pairwise orthogonal vectors in \mathcal{H} . TFAE :

$$(1) \sum_{n=1}^{\infty} x_n \text{ converges in } \mathcal{H}.$$

$$(2) \sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$$

$$(3) \sum_{n=1}^{\infty} \langle x_n, y \rangle \text{ converges } \forall y \in \mathcal{H}.$$

Proof :

We have for all n, m with $1 \leq n \leq m$,

$$\|x_n + \dots + x_m\|^2 = \|x_n\|^2 + \dots + \|x_m\|^2$$

Hence (2) implies that the partial sums of $\sum_{n=1}^{\infty} x_n$ form a Cauchy

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sequence, hence the series converges
which shows (2) \Rightarrow (1).

Next, Cauchy-Schwarz implies:

$$\left| \sum_{k=n}^m \langle x_k, y \rangle \right| = \left| \langle \sum_{k=n}^m x_k, y \rangle \right| \\ \leq \left\| \sum_{k=n}^m x_k \right\| \|y\|$$

and hence (1) implies (3).

Now assume (3) and define $\lambda_n \in \mathcal{H}^*$
by $\lambda_n(y) := \sum_{i=1}^n \langle y, x_i \rangle$.

Then (3) implies $\lim_{n \rightarrow \infty} \lambda_n(y)$ exists

for every $y \in \mathcal{H}$ which by Banach-
Steinhaus implies that $\{\|\lambda_n\|_{\mathcal{H}^*}\}$

is bounded. But

$$\|\lambda_n\| = \|x_1 + \dots + x_n\| = \sqrt{(\|x_1\|^2 + \dots + \|x_n\|^2)} \\ \text{which implies (2). } \blacksquare$$

Proof of Prop. 5.13

We have seen that $E_{x,y}(\omega) = \langle E(\omega)x, y \rangle$

ω is complex measure and hence

with $\omega = \bigsqcup_{n \geq 1} \omega_n$ as in the statement

of Prop. 5.13 we have

$$E_{x,y}(\omega) = \sum_{n=1}^{\infty} E_{x,y}(\omega_n)$$

that is $\langle E(\omega)x, y \rangle = \sum_{n=1}^{\infty} \langle E(\omega_n)x, y \rangle$.
(*)

Now since $\forall n \neq m \quad \omega_n \cap \omega_m = \emptyset$

we have $E(\omega_n)x \perp E(\omega_m)x$. Thus

Lemma 5.14 implies that

$$\sum_{n=1}^{\infty} E(\omega_n)x$$

converges in \mathcal{H} and as a result

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$$(*) \text{ implies } \langle E(w)x, y \rangle = \left\langle \sum_{n=1}^{\infty} E(w_n)x, y \right\rangle$$

$\forall y \in \mathcal{H}$, hence:

$$E(w)x = \sum_{n=1}^{\infty} E(w_n)x.$$



Let us observe that since for a self-adjoint projection P we have $\|P\| = 0$ or 1, the series $\sum_{n \geq 1} E(w_n)w_n'$ not converge in $L(\mathcal{H})$ unless all but finitely many $E(w_n)'s are zero.$

5.5. The Algebra $L^\infty(E)$

Let X be a compact Hausdorff space and $E : \mathcal{B} \rightarrow \mathcal{L}(f\epsilon)$ a resolution of the identity. We proceed to define the C^* -algebra $L^\infty(E)$ of ~~closed~~ of bounded Borel functions.

Let $f : X \rightarrow \mathbb{C}$ be a measurable function; we proceed to define the essential range of f . This will use the following

Lemma 5.15 Let $w = \bigcup_{n \geq 1} w_n$ where $w_n \in \mathcal{B}$ and $E(w_n) = 0$. Then $E(w) = 0$.

Prof: Since $E(w_n) = 0$ we have

$$E_{x,x}(w_n) = 0 \quad \forall x \in X. \text{ Now } E_{x,x}$$

is σ -additive, hence $E_{x,x}(w) = 0 \quad \forall x \in \mathbb{C}$
which implies $E(w) = 0$. \square

Let then $\{D_n : n \geq 1\}$ be a countable collection of open discs in \mathbb{C} forming a basis for the topology of \mathbb{C} . Define

$$V := \bigcup \{ D_n : E(\bar{f}'(D_n)) = 0 \}$$

Then V is open and it follows from Lemma 5.15 that $E(\bar{f}'(V)) = 0$.

Clearly V is the largest open set with this property. We call the closed set $\mathbb{C} \setminus V$ the essential range of f denoted $\text{Ess Im}(f)$. We say f is essentially bounded if $\text{Ess Im}(f)$ is bounded and define

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$$\|f\|_\infty = \sup \{|x| : x \in E \text{ s.t. } f(x) \neq 0\}.$$

Next let $B^\infty(X)$ denote the space of bounded Borel functions with the norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Then one verifies that $B^\infty(X)$ is a C^* -algebra, in fact an abelian one.

The subspace

$$N = \{f \in B^\infty(X) : \|f\|_\infty = 0\}$$

is then a closed ideal in $B^\infty(X)$

(Exercise) and we define

$$L^\infty(E) := B^\infty(X) / N.$$

Then the quotient norm of a class $[f] = f + N$ is just $\|f\|_\infty$.