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and hence $\mu_f = \int_0^1 dw$ which proves
the inversion formula. \square

We turn to an important consequence
of the inversion theorem.

Recall from chapter 7 that for
 $C \subset \widehat{G}$ compact and $r > 0$ we
have defined

$$N(C, r) = \{x \in G : |(x, y)^{-1}| < r \\ \forall y \in C\}$$

and have shown that it is an open
set; in fact it is an open neighborhood

of $e \in G$. Now we show

Corollary 9.11 $\{g \cdot N(C, r) : g \in G,$
 $C \subset G \text{ compact}, r > 0\}$ is a basis
of the topology of G .

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Proof: Let $V \ni e$ be an open neighborhood; let $W \ni e$ be a compact neighborhood of e with $WW^{-1} \subset V$. Define

$f = X_w / \lambda(w)$ and set

$$g = f * f^*$$

The strategy is to find a compact $G \subset \hat{G}$ such that $\forall x \in N(c, \gamma_3)$,

$$g(x) > 0$$

which will imply $x \in WW^{-1} \subset V$

and hence $N(c, \gamma_3) \subset V$.

To this end we apply the inversion theorem to g , which we may since

$$\cancel{g \in L^1(G)} \quad g \in C_0(G) \cap P(G).$$

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$$\text{So } g(x) = \int_{\hat{G}} \hat{g}(\gamma)(x, \gamma) d\omega(\gamma). \quad (*)$$

First, evaluate at $x = e$; observe

$$g(e) = 1 \text{ and } |\hat{f}(\gamma)| = |\hat{f}(\gamma)|^2 \geq 0 \forall \gamma.$$

Thus $1 = \int_{\hat{G}} \hat{g}(\gamma) d\omega(\gamma)$

and there is thus $\hat{G} \subset \hat{\mathbb{C}}$ compact

with

$$\int_{\hat{G}} \hat{g}(\gamma) d\omega(\gamma) > \frac{2}{3}.$$

Let now $x \in N(G, \gamma_3)$. Write (*)

$$g(x) = \int_G \hat{g}(\gamma)(x, \gamma) d\omega(\gamma) + \int_{\hat{G} \setminus G} \hat{g}(\gamma)(x, \gamma) d\omega(\gamma)$$

Now $\left| \int_{\hat{G} \setminus G} \hat{g}(\gamma)(x, \gamma) d\omega(\gamma) \right| \leq \int_{\hat{G} \setminus G} |\hat{g}(\gamma)| d\omega(\gamma)$
 $< \frac{1}{3}.$

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Also

$$g(x) = \int_C \hat{g}(\gamma) \operatorname{Re}(x, \gamma) d\omega(\gamma) + \operatorname{Re}\left(\int_{G \setminus C} \hat{g}(\gamma) d\omega(\gamma)\right)$$

Observe that if $|1 - (x, \gamma)| < \gamma_3$ then

$\operatorname{Re}(x, \gamma) > \frac{2}{3}$ and hence

$$\int_C \hat{g}(\gamma) \operatorname{Re}(x, \gamma) d\omega(\gamma) > \frac{2}{3} \int_C \hat{g}(\gamma) d\omega(\gamma) > \frac{4}{9}.$$

Thus $g(x) > \frac{4}{9} - \frac{1}{3} = \frac{1}{9}$. \blacksquare

Corollary 9.12 \hat{G} separates points in G ,
that is $\forall x_1 \neq x_2$ in G there is
 $x \in \hat{G}$ with $x(x_1) \neq x(x_2)$.

Proof: $\bar{x}_1'x_2 \neq e$; let V be open such that $V \ni e$ and $\bar{x}_1'x_2 \notin V$ and let $G \subset \hat{G}$ be compact such that $N(G, v_3) \subset V$. Then :
 $\bar{x}_1'x_2 \notin N(c, v_3)$ hence there is $x \in G$ with $|X(\bar{x}_1'x_2) - 1| \geq v_3$ which concludes the proof. \square

Examples 9.13

(1) $G = \mathbb{R}$: let $X(t) = e^{2\pi i t}$

and identify $\hat{\mathbb{R}}$ with \mathbb{R} via

$$\mathbb{R} \rightarrow \hat{\mathbb{R}}$$

$$a \mapsto x_a$$

where $x_a(t) = \chi(a + t)$. Let \mathcal{L} be
the Lebesgue measure on \mathbb{R} . Then

$$\hat{f}(a) = \int_{\mathbb{R}} f(t) e^{-2\pi i a t} d\mathcal{L}(t)$$

and

$$f(t) = \int_{\mathbb{R}} \hat{f}(a) e^{2\pi i a t} d\mathcal{L}(a)$$

provided $f \in L^1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$.

(2) $G = \mathbb{Q}_p$: we fix a continuous character $\chi: \mathbb{Q}_p \rightarrow \overline{\mathbb{U}}$ with $\text{Ker } \chi = \mathbb{Z}_p$, which is possible since the discrete group $\mathbb{Q}_p / \mathbb{Z}_p$ is isomorphic to the subgroup of $\overline{\mathbb{U}}$:

$$\left\{ z \in \overline{\mathbb{U}} : \exists n \geq 1 \text{ with } z^{p^n} = 1 \right\}.$$

Now identify $\widehat{\mathbb{Q}_p}$ with \mathbb{Q}_p via

$$\mathbb{Q}_p \xrightarrow{\sim} \widehat{\mathbb{Q}_p}$$

$$a \mapsto x_a, \quad x_a|t_1 = x|t_1.$$

Let λ be the Haar measure on \mathbb{Q}_p such that $\lambda(\mathbb{Z}_p) = 1$. One computes then

$$\widehat{\chi}_{\mathbb{Z}_p}(a) = \begin{cases} 1 & \text{if } x_a|_{\mathbb{Z}_p} \equiv 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now $\chi_a|_{\mathbb{Z}_p} = 1$ if $a \in \mathbb{Z}_p$

$\forall t \in \mathbb{Z}_p$ iff $t \in \mathbb{Z}_p$. Thus:

$$\widehat{\chi}_{\mathbb{Z}_p} = \chi_{\mathbb{Z}_p}.$$

Thus λ is also the right normalized Haar measure for the inversion formula to hold.

(3) If G is compact we have seen that \widehat{G} is discrete. Let λ be the Haar measure on G s.t. $\lambda(G) = 1$.

We have seen that then

$$\widehat{\mathbf{1}}_G(x) = \begin{cases} 1 & x = \delta_{\hat{e}} \\ 0 & x \neq \hat{e} \end{cases}$$

and hence $\mathbf{1}_G(e) = \sum_{y \in \widehat{G}} \delta_{\hat{e}}(y)$

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which implies that the dual Haar measure w on \widehat{G} is counting measure.

§. 3. The Plancherel Theorem.

In this section we will prove Plancherel's Theorem and draw some consequences among which a characterisation of $\mathcal{A}(\widehat{G})$ in terms of $L^2(\widehat{G})$. We fix Haar measures λ on G and μ on \widehat{G} such that the inversion theorem (Thm 3.8) holds.

Theorem 3.14 (Plancherel)

The Fourier transform

$$\begin{aligned} L^1(G) \cap L^2(G) &\longrightarrow \mathcal{C}_0(\widehat{G}) \\ f &\mapsto \widehat{f} \end{aligned}$$

extends to an isometric isomorphism

$$L^2(G) \rightarrow L^2(\widehat{G})$$

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Proof:

Let $f \in L^1(G) \cap L^2(G)$: then $f * f^* \in L^1(G)$, and it is also in $\mathcal{P}(G)$ as observed in the proof of the inversion theorem (see 9-24). Thus $g := f * f^* \in L^1(G) \cap \mathcal{P}(G)$ and the inversion theorem holds. In particular:

$$g(c) = \int_{\widehat{G}} \widehat{g}(x) d\omega(x). \quad (*)$$

$$\begin{aligned} \text{Now } g(c) &= f * f^*(c) = \int_G f(x) \overline{f^{(n)}(x)} d\lambda(x) \\ &= \|f\|_2^2 \end{aligned}$$

$$\text{and } \widehat{g}(x) = \widehat{f}(x) \cdot \overline{\widehat{f}(x)}$$

which implies with (*) that

$$\|f\|_2^2 = \|\widehat{f}\|_2^2.$$

It remains to show that the subspace

$$\mathcal{L}' = \left\{ \hat{f} : f \in L^1(G) \cap L^2(G) \right\}$$

is dense in $L^2(\hat{G})$. To this effect

let $\psi \in L^2(\hat{G})$ with:

$$\int_{\hat{G}} \hat{f}(y) \overline{\psi(y)} d\omega(y) = 0$$

$\forall f \in L^1(G) \cap L^2(G)$. Since $L^1(G) \cap L^2(G)$

is translation invariant we have

that $\forall x \in G$, $y \mapsto (x, y) \hat{f}(y)$ is
in \mathcal{L}' as well, hence

$$\int_{\hat{G}} \hat{f}(y) \overline{\psi(y)} (x, y) d\omega(y) = 0$$

Now $\hat{f} \cdot \overline{\psi} \cdot \omega \in M(\hat{G})$ and hence

by the injectivity part of Coroll. 9.7

we get $\hat{f} \cdot \overline{\psi} \cdot \omega = 0$, hence

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for every $f \in L^1(\mathbb{C}) \cap L^2(G)$ we have

$$\hat{f} \cdot \gamma = 0 \text{ almost everywhere.}$$

Now by Lemma 9.10 we can find

for every $V \subset \hat{G}$ open with \bar{V} compact

an $f \in C_0(G)$ with $\hat{f} > 0$ on V .

Hence $\gamma = 0$ almost everywhere and the theorem is proven.



In the sequel we will denote again by \hat{f} the function in $L^2(\hat{G})$ corresponding to $f \in L^1(\mathbb{C})$ by the Plancherel theorem. Care must be taken that if $f \notin L^1(G)$, \hat{f} is not given by the familiar integral formula!

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Corollary 9.15 $\forall f, g \in L^2(G)$

we have

$$\int_G f(x) \overline{g(x)} d\mu(x) = \int_{\widehat{G}} \widehat{f}(y) \overline{\widehat{g}(y)} d\nu(y).$$

Proof: This follows from the fact that
a linear isometry between Hilbert
spaces preserves the scalar product.

□

Theorem 9.16

$$A(\widehat{G}) = \left\{ F_1 * F_2 : F_1, F_2 \in L^2(\widehat{G}) \right\}.$$

Proof:

Recall from Thm 6.16 (3) with $p=2$

that for every $F_1, F_2 \in L^2(\widehat{G})$,

$F_1 * F_2$ is well defined and in $C_0(\widehat{G})$.

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$$\int_{\widehat{G}} \widehat{f}(\gamma) \widehat{g}(\gamma, \gamma') d\omega(\gamma) = \widehat{f} * \widehat{g}(\gamma_0).$$

That is $(\widehat{f} \cdot \widehat{g})(\gamma_0) = \widehat{f} * \widehat{g}(\gamma_0)$ (*)

Applying Plancherel's theorem to the
fact that the Fourier transform is

an isomorphism $L^2(G) \xrightarrow{\sim} L^2(\widehat{G})$

and in particular surjective, this
shows that $\forall F_1, F_2 \in L^2(\widehat{G})$,

$$F_1 * F_2 \in A(\widehat{G}).$$

Conversely : let $h \in L^1(G)$; then

$$h(x) = \langle h(x), \psi(x) \rangle \text{ with}$$

$$\psi(x) = \begin{cases} h(x)/|h(x)| & \text{if } h(x) \neq 0 \\ 1 & \text{if } h(x) = 0. \end{cases}$$

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Now let $f, g \in L^2(G)$; we want to compute

$$\int_G f(x) g(x) (\bar{x}, \delta_0) d\lambda(x)$$

using Parseval's formula; to this end let $u(x) = \overline{g(x)} (\bar{x}, \delta_0)$, then we need to compute $\overline{\hat{u}}$. This is straightforward if in addition $g \in L^1(G)$ as a direct computation gives:

$$\overline{\hat{u}}(\delta) = \hat{g}(\delta, \delta')$$

Thus (exercise) this formula holds for a.a. $g \in L^2(G)$. Parseval's formula (Cor. 9.15) then gives

$$\int_G f(x) g(x) \overline{(x, \delta_0)} d\lambda(x) =$$

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Then $f(x) := |h(x)|^{\frac{1}{12}}$ and $g(x) = h(x)/f(x)$

satisfy $h = f \cdot g$ with $f, g \in L^2(G)$.

The above formula then gives

$$\hat{h} = \hat{f} * \hat{g}$$

which shows $A(\hat{G}) \subset \{F_1 * F_2 : F_i \in L^2(\hat{G})\}$

and concludes the proof of the theorem.

□

The following consequence will be used
in the Pontryagin duality theorem.

Prop. 5.17 Let $E \subset \hat{G}$ be non-empty
open. Then there is $\gamma \in A(\hat{G})$ with
 $\gamma \neq 0$ and $\gamma|_{\hat{G} \setminus E} = 0$.

Proof: Pick any $\delta_0 \in E$; using
the continuity at $(\delta_0, \hat{\epsilon}) \in \hat{G} \times \hat{G}$

of the product we can find compact neighborhoods $K \ni x_0$ and $F \ni e$ such that $K \cdot F \subset E$. Then

$$\psi := \underset{K}{\chi} * \underset{F}{\chi} \in A(\hat{G})$$

and $\psi \neq 0$. Since $\text{supp}(\psi) \subset K \cdot F$ we have $\psi|_{\hat{G} \setminus E} = 0$.

