

Let A be a commutative Banach algebra.

Def. 3.9 A character of A is a

\mathbb{C} -algebra homomorphism $\chi: A \rightarrow \mathbb{C}$

that is not identically zero.

We denote by \hat{A} the set of characters of A , it is called the Gelfand spectrum of A .

It will be crucial to establish the relation between \hat{A} and $\hat{A}_{\mathbb{I}}$.

Remark 3.10

Every $\varphi \in \hat{A}$ has a unique extension

$\tilde{\varphi}$ to $A_{\mathbb{I}} = A \times \mathbb{C}$ given by

$$\tilde{\varphi}(x, \lambda) = \varphi(x) + \lambda$$

Let $\tilde{\hat{A}} = \{ \tilde{\varphi} : \varphi \in \hat{A} \} \subset \hat{A}_{\mathbb{I}}$.

If $\gamma_{\infty}: A_{\mathbb{I}} \rightarrow \mathbb{C}$ denotes the

the character given by $\varphi_\infty(x, \lambda) = \lambda$

then $\hat{A}_I = \tilde{A} \cup \{\varphi_\infty\}$.

In the sequel we will whenever convenient identify \hat{A} with \tilde{A} and drop the latter notation.

The following proposition is a little miracle in the theory of abelian

Banach algebras:

Prop. 3.11 For every $\varphi \in \hat{A}$ we have

$$|\varphi(x)| \leq \|x\|_{sp} \quad \forall x \in A.$$

In particular φ is a bounded linear functional with $\|\varphi\| \leq 1$ in general

and $\|\varphi\| = 1$ if A is unital.

Proof: Because of remark 3.10 we may

assume A is unital.

If $|\lambda| > \|x\|_{sp}$ then $x - \lambda e$ is invertible and since $\varphi(e) = 1$ we conclude $\varphi(x) - \lambda \neq 0$. Hence

$$|\varphi(x)| \leq \|x\|_{sp} \leq \|x\| \quad \forall x.$$

If A is unital, $\varphi(e) = 1$ hence $\|\varphi\| = 1$.



Example 3.12 We have seen in the exercises that if $A = L^1([0, 1])$ is the Volterra algebra then

$$\|f\|_{sp} = 0 \quad \forall f \in A.$$

Hence $\hat{A} = \phi$.

The next result gives the relationship announced at the beginning of section 3 between characters and maximal regular ideals.

Thm 3.13 For a commutative Banach algebra the mapping

$$\varphi \mapsto \text{Ker } \varphi$$

establishes a bijection between \hat{A} and the set $\text{Max}(A)$ of maximal regular ideals in A .

Proof:

(1) For $\varphi \in \hat{A}$, $\text{Ker } \varphi$ is an ideal and it is maximal because it is a codimension 1 subspace of A . Moreover it is regular by Example 3.2.

(2) If $\text{Ker } \varphi_1 = \text{Ker } \varphi_2 := I$ let $u \in A$ be an identity modulo I .

Since $I + \mathbb{C}u = A$, let $x = x_0 + \lambda u$.

Then $\varphi_1(x) = \lambda \varphi_1(u) = \lambda = \lambda \varphi_2(u) = \varphi_2(x)$.

(3) Let $I \in \text{Max}(A)$ and $u \in A$

an identity modulo I . By Corollary 3.4

I is closed and hence by Prop. 3.7

A/I is a Banach algebra wrt the quotient algebra structure. Clearly

A/I is unital with unite $u + I$.

We claim that every $x + I$ with $x \notin I$

is invertible. If not, then ~~by Corollary 3.6~~

$(x + I) \cdot A/I := J$ is a proper ideal

of A/I and hence its inverse image J'

in A is an ideal satisfying

$$I \subsetneq J' \subsetneq A$$

contradicting the maximality of I .

Hence by the Gelfand-Mazur theorem

A/I is isomorphic to \mathbb{C} , which

leads to a character $\chi \in \hat{A}$ with $\text{Ker } \chi = I$. □

Example 3.14 (see Prop. 3.8)

Let X be l.c. Hausdorff. Then

$\forall x \in X$, the evaluation $\varphi \mapsto \varphi(x)$

defines a character of $C_0(X)$ which

leads to a bijection

$$X \rightarrow \widehat{C_0(X)}.$$

Let then $\text{Rad}(A) = \bigcap \{M : M \in \text{Max}(A)\}$

$$= \bigcap \{ \text{Ker } \varphi : \varphi \in \hat{A} \}$$

Def. 3.15 $\text{Rad}(A)$ is the radical of A .

And A is called semisimple if $\text{Rad}(A) = (0)$.

The automatic continuity of characters

has then consequences for \mathbb{C} -algebra

homomorphisms with values in semi-simple commutative Banach algebras.

Corollary 3.16 Let $\varphi: A \rightarrow B$ be a \mathbb{C} -algebra homomorphism with A, B commutative Banach algebras and B semi-simple. Then φ is continuous.

~~Proof:~~

Recall:

Closed Graph Thm. $T: E \rightarrow F$ linear map of Banach spaces. TFAE:

- (i) T is continuous.
- (ii) $\text{Graph}(T) \subset E \times F$ is closed.
- (iii) If $x_n \rightarrow 0$ in E and $Tx_n \rightarrow y$ in F , then $y = 0$.

Proof of Cor. 3.16: Let $(x_n)_{n \geq 1}$ in A with $x_n \rightarrow 0$ and $\varphi(x_n) \rightarrow b$ in B .

Let $\chi \in \hat{B}$; then $\chi \circ \varphi \in \hat{A} \cup \{0\}$ and both χ and $\chi \circ \varphi$ are continuous.

Thus:

$$\begin{aligned} \chi(b) &= \lim_{n \rightarrow \infty} \chi(\varphi(x_n)) = \lim_{n \rightarrow \infty} (\chi \circ \varphi)(x_n) \\ &= (\chi \circ \varphi)(0) = 0 \end{aligned}$$

and since B is semisimple, $b = 0$.

□

We have seen in Example 1.7 (3) that for every $n \geq 0$, $C^n([0, 1])$ admits a Banach algebra norm. One can use Corollary 3.16 to show:

Example 3.17 $C^\infty([0, 1])$ does not admit any Banach algebra norm.

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The idea is the following: assume $\|\cdot\|$ is a Banach algebra norm on $C^\infty([0,1])$. The algebra $C([0,1])$ with $\|\cdot\|_\infty$ is semisimple and the inclusion $C^\infty([0,1]) \hookrightarrow C([0,1])$ is hence continuous; thus there is $c > 0$ with $\|f\|_\infty \leq c \|f\| \forall f \in C^\infty([0,1])$. Use this inequality and the closed graph theorem to show that the derivative $D: f \mapsto f'$ is continuous. Reach a contradiction by using the functions $t \mapsto \exp(\alpha t)$.

Example 3.8: Consider $A = \ell^2(\Gamma)$

where Γ is an abelian group; for example $\Gamma = \text{finite}$, \mathbb{Z}^n , \mathbb{R}_S that is \mathbb{R} with discrete topology. Let's compute \hat{A} .

Let $\chi \in \widehat{\ell^1(\Gamma)}$; in particular $\chi \in \ell^1(\Gamma)^*$
 $= \ell^\infty(\Gamma)$, that is, there exists a unique
 $H \in \ell^\infty(\Gamma)$ with

$$\chi(f) = \sum_{\gamma} f(\gamma) H(\gamma) \quad \forall f \in \ell^1(\Gamma).$$

Evaluating χ on δ_γ gives $\chi(\delta_\gamma) = H(\gamma)$

And from the multiplicativity of χ

we get =

$$\begin{aligned} H(\gamma_2) &= \chi(\delta_{\gamma_2}) = \chi(\delta_\gamma * \delta_\gamma) \\ &= \chi(\delta_\gamma) \chi(\delta_\gamma) \\ &= H(\gamma) H(\gamma). \end{aligned}$$

Also $H(e) = \chi(\delta_e) = 1$ and

from $\|H\|_\infty = \|\chi\| = 1$ we deduce

$|H(\gamma)| = 1 \quad \forall \gamma \in \Gamma$. Thus let

$$\mathbb{T} := \{ z \in \mathbb{C}^* : |z| = 1 \}$$

we have obtained the identifications:

$$\widehat{\ell^1(\Gamma)} \cong \text{Hom}(\Gamma, \Pi).$$

We will see later that $\ell^1(\Gamma)$ is semi-simple; but this fact lies deeper.

Now we move to a crucial point namely that the Gelfand spectrum has a natural topology. One approach would be to observe that \hat{A} is contained in the unit ball of the dual A^* of A and restrict the weak $*$ -topology to \hat{A} . We are going to choose a self-contained approach and only make use of Tychonov's theorem.