

Remark 6.11. In the preceding example

we have used that if μ is a left H -invariant measure on G then for any non-empty open $V \subset G$, $\mu(V) > 0$.

Indeed: let $f \in C_0(G)$ with $f \geq 0$

such that $\int_G f(x) d\mu(x) > 0$.

Since $\text{supp } f$ is compact we may find x_1, \dots, x_n in G such that

$$\text{supp } (f) \subset \bigcup_{i=1}^n x_i \cdot V$$

If $M = \max \{ f(x) : x \in G \}$ then we

have:

$$f(x) \leq M \cdot \sum_{i=1}^n \chi_{x_i \cdot V}(x) \quad \forall x \in G$$

and hence

$$\begin{aligned} 0 < \int_G f(x) d\mu(x) &\leq \int_G M \cdot \sum_{i=1}^n \chi_{x_i \cdot V}(x) d\mu(x) \\ &= M \cdot \sum_{i=1}^n \mu(x_i \cdot V) = \end{aligned}$$

$$= M \cdot n \cdot \mu(V) \quad \text{which implies } \mu(V) > 0.$$

We can apply Corollary 6.9 to a special class of automorphisms of G namely the inner automorphisms:

$$\alpha_g(x) := g x g^{-1}, \quad x \in G.$$

Then if μ is a left Haar measure the equality in Cor. 6.9 reads

$$\int_G f(g^{-1} x g) d\mu(x) = \text{mod}_G(\alpha_g) \int_G f(x) d\mu(x)$$

Which taking into account the left invariance of μ implies

$$\int_G f(xg) d\mu(x) = \text{mod}_G(\alpha_g) \int_G f(x) d\mu(x).$$

Definition 6.11 The modular function of G is $\Delta_G(g) := \text{mod}_G(\alpha_{g^{-1}})$

So that: $\int_G f(xg^{-1}) d\mu(x) = \Delta_G(g) \int_G f(x) d\mu(x)$.

Then one can show

Prop. 6.12 $\Delta_G : G \rightarrow \mathbb{R}_{>0}$ is a

continuous homomorphism and

$$\int_G f(x^{-1}) \Delta_G(x^{-1}) d\mu(x) = \int_G f(x) d\mu(x).$$

Observe that if G is abelian $\Delta_G \equiv 1$.

When G is abelian, Prop. 6.12 reads

$$\int_G f(x^{-1}) d\mu(x) = \int_G f(x) d\mu(x)$$

and can be proven as follows:

define $\Lambda(f) := \int_G f(x^{-1}) d\mu(x)$.

$$\begin{aligned} \text{Then } \Lambda(\int_G f) &= \int_G (\int_G f)(x^{-1}) d\mu(x) \\ &= \int_G \int_G f(g^{-1}x^{-1}) d\mu(x) = \int_G \int_G f(xg^{-1}) d\mu(x) \\ &= \int_G \int_G f(xg^{-1}) d\mu(x) \quad \text{since } G \text{ is} \end{aligned}$$

abelian; the latter then equals

$$= \int_G \int_G f(x^{-1}) d\mu(x) \quad \text{by invariance.}$$

Thus there is $c > 0$ such that

$$\int_G \int_G f(x^{-1}) d\mu(x) = c \int_G f(x) d\mu(x)$$

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$\forall f \in C_0(\mathbb{R})$. If now $f \geq 0$,

$f \in C_0(\mathbb{R})$ with $\int_{\mathbb{R}} f(x) d\mu(x) \geq 0$

then

$$g(x) := f(x) + f(x^{-1})$$

satisfies $g(x^{-1}) = g(x)$ and

$$\int_{\mathbb{R}} g(x^{-1}) d\mu(x) = c \cdot \int_{\mathbb{R}} g(x) d\mu(x)$$

$$= \int_{\mathbb{R}} g(x) d\mu(x)$$

hence $c = 1$.

6.4. The convolution product.

For a locally compact group G and a left Haar measure μ we denote by $L^p(G) := L^p(G, \mu)$ the usual L^p -spaces associated to μ .

We are defining the convolution product of two functions on G and show in particular that $L^1(G)$ is a involutive Banach algebra. All the results stated here hold for general locally compact Hausdorff groups, but we will present the proofs for σ -compact ~~pro~~ groups, that is groups that are a countable

union of compact subsets. In this case one can use standard versions of Fubini's theorem as the measure space (G, μ) is σ -finite; for later use we also mention that if $1 \leq p < +\infty$ and $\frac{1}{q} + \frac{1}{p} = 1$, then $L^q(G)$ is the Banach space dual to $L^p(G)$; this applies in particular to $L^1(G)$ whose dual is then $L^\infty(G)$.

Let $f, g \in L^1(G)$. Applying Fubini's theorem to the positive measurable function

$$\begin{aligned} G \times G &\longrightarrow [0, \infty] \\ (x, y) &\longmapsto (f(xy)g(y^{-1})) \end{aligned}$$

we have

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$$\begin{aligned} & \int_G d\mu(x) \int_G d\mu(y) |f(xy)| |g(y')| \\ &= \int_G d\mu(y) \int_G d\mu(x) |f(xy)| |g(y')| \\ &= \int_G d\mu(y) |g(y')| |\Delta_G(y')| \int_G d\mu(x) |f(x)| \\ &= \|g\|_1 \cdot \|f\|_1 < +\infty. \end{aligned}$$

By Lebesgue's theorem we conclude that the function

$$x \mapsto \int_G d\mu(y) |f(xy)| |g(y')|$$

is measurable, almost everywhere finite and so is:

$$f * g(x) := \int_G d\mu(y) f(xy) g(y').$$

By the above computation we get

$$\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1.$$

~~Thus.~~

Define now $f^*(x) := \overline{f(x^{-1})} \Delta_G(x^{-1})$.

Then

Prop. 6.13 $L^1(G)$ endowed with the convolution product $f * g$ for $f, g \in L^1(G)$ and $f \mapsto f^*$ is an involutive Banach algebra.

Proof. It remains to show

(1) that the convolution product is associative.

(2) $f \mapsto f^*$ satisfies all the properties of Def. 1.7. We leave these verifications to the reader. \square

In order to develop Fourier Analysis we will need some additional continuity properties of the convolution product.

We begin with a definition:

Definition 6.14 Let (X, d) be a metric space. A function $F: G \rightarrow X$ is left uniformly continuous if $\forall \epsilon > 0$ there is a neighborhood $V \ni e$ of e such that: $d(F(x), F(y)) < \epsilon$ whenever $y^{-1}x \in V$.

The terminology comes from the fact that

if $y^{-1}x \in V$ then $(gy)^{-1}(gx) = y^{-1}x \in V$
 $\forall g \in G.$

Lemma 6.15

$$(1) \forall f \in C_0(G), \quad G \rightarrow C_0(G) \\ x \mapsto \lambda(x)f$$

is left uniformly continuous wrt the $\|\cdot\|_\infty$ -norm.

$$(2) \text{ Let } 1 \leq p < \infty; \text{ then } \forall f \in L^p(G)$$

$x \mapsto \lambda(x)f$ is left uniformly cont.

wrt $\|\cdot\|_p$ -norm.

Proof:

(1) Since $\|\cdot\|_\infty$ is left-translation invariant we have $\forall f \in C_0(G), \forall x, y \in G$

$$\|\lambda(x)f - \lambda(y)f\|_\infty = \|\lambda(y^{-1}x)f - f\|_\infty$$

Since $(\lambda(y^{-1}x)f)(g) = f(y^{-1}xg)$ we have

to show that $\forall \varepsilon > 0, \exists W \ni e$ open

with $|f(yg) - f(g)| < \varepsilon \quad \forall g \in G, \forall y \in W$.

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Fix $f \in C_0(G)$; let $K = \text{supp}(f)$

and pick $V_0 = V_0^{-1} \ni e$ open with

$\overline{V_0}$ compact. Let $\epsilon > 0$; for every

$x \in G$ let $V_x \ni e$ open with $V_x \subset V_0$

such that

$$|f(\beta x) - f(x)| < \epsilon/2 \quad \forall \beta \in V_x.$$

Choose $U_x \ni e$ open with $U_x \cdot U_x \subset V_x$.

Since $\overline{V_0} \cdot K$ is compact we may

find x_1, \dots, x_n in G such that

$$\overline{V_0} \cdot K \subset \bigcup_{i=1}^n U_{x_i}$$

Define $W := \bigcap_{i=1}^n U_{x_i} \subset V_0$.

(1) if $g \notin \overline{V_0} \cdot K$ then $\forall \beta \in W$,

$\beta g \notin K$ and hence $f(\beta g) = f(g) = 0$

(2) if $g \in \overline{V_0} \cdot K$ and $\beta \in W$ let

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$\exists \delta > 0$ such that $g \in U_{x_i}$;

then $\exists g \in V_{x_i}$ and hence

$$|f(3g) - f(g)| \leq |f(3g) - f(x_i)| + |f(x_i) - f(g)| \\ < \epsilon/2 + \epsilon/2 = \epsilon.$$

(2) We use that since $1 \leq p < +\infty$

$C_0(G)$ is dense in $L^p(G)$ wrt $\|\cdot\|_p$.

Let then $f \in L^p(G)$, $\epsilon > 0$ and

$g \in C_0(G)$ with $\|f - g\|_p < \epsilon$.

Then

$$\|\lambda(x)f - \lambda(y)f\|_p \leq \|\lambda(x)f - \lambda(x)g\|_p + \|\lambda(x)g - \lambda(y)g\|_p \\ + \|\lambda(y)g - \lambda(y)f\|_p$$

$$< 2\epsilon + \|\lambda(\bar{y}^{-1}x)g - g\|_p.$$

Pick $V_0 = V_0^{-1}$ open with \bar{V}_0 compact.

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We may assume $\bar{y}'x \in V_0$. Then

$$\forall t \notin K := \bar{V}_0 \cdot \text{supp}(g)$$

$$\lambda(\bar{y}'x)g(t) - g(t) = 0$$

Hence

$$\|\lambda(\bar{y}'x)g - g\|_p = \left[\int_K |\lambda(\bar{y}'x)g(t) - g(t)|^p d\mu(t) \right]^{1/p}$$

$$\leq \|\lambda(\bar{y}'x)g - g\|_\infty \cdot \mu(K)^{1/p}$$

Now choose $\varepsilon \in W \subset V_0$ open such that

$$\|\lambda(\bar{y}'x)g - g\|_\infty < \frac{\varepsilon}{\mu(K)^{1/p}}$$

$\forall \bar{y}'x \in W$. Then we conclude

$$\|\lambda(x)g - \lambda(y)g\|_p < 3\varepsilon.$$



Theorem 6.16.

Let G be an abelian locally compact Hausdorff group.

(1) $f \in L^1(G)$, $g \in L^\infty(G)$ then $f * g$ is bounded and uniformly continuous.

(2) $f, g \in C_0(G)$ then $f * g \in C_0(G)$
and $\text{supp}(f * g) \subset \text{supp}(f) \text{supp}(g)$

(3) $1 < p < +\infty$, $f \in L^p(G)$ and $g \in L^q(G)$ then $f * g \in C_0(G)$.

Proof:

(1) We have

$$|f * g(x)| = \left| \int_G f(xy) g(y^{-1}) d\mu(y) \right| \leq \|f\|_1 \|g\|_\infty$$

$\forall x \in G.$

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For $x, z \in G$ we have:

$$|f * g(x) - f * g(z)| \leq \int_G (|f(xy) - f(zy)| |g(y')|) dy$$

$$\leq \| \lambda(x) f - \lambda(z) f \|_1 \|g\|_\infty$$

Which together with lemma 6.15 shows
(1).

(2) We already know that $f * g$ is continuous by (1). If $f * g(x) \neq 0$ there is $y \in G$ with $f(xy) \neq 0$ and $g(y') \neq 0$. Hence $xy \in \text{supp } f$, $y' \in \text{supp } g$ which with $x \in (\text{supp } f) \setminus y'$ implies $x \in (\text{supp } f) \cap \text{supp } g$.

(3) Observe first that if $f \in L^1(G)$ and $g \in L^1(G)$ then

Since G is abelian, $y \mapsto g(y')$ is in $L^q(G)$; it follows then from Hölder's inequality that $\forall x \in G$,

$$y \mapsto f(xy)g(y')$$

is in $L^1(G)$ and $f * g$ is defined.

Next we use that since $1 \leq p < \infty$

$C_{00}(G)$ is dense in $L^p(G)$ and $L^q(G)$.

Choose sequences $(f_n)_{n \geq 1}$ and $(g_n)_{n \geq 1}$

in $C_{00}(G)$ with $\|f - f_n\|_p \rightarrow 0$ and

$$\|g - g_n\|_q \rightarrow 0.$$

Then

$$\begin{aligned} |f * g(x) - f_n * g_n(x)| &\leq |(f - f_n) * g_n(x)| + \\ &\quad + |f_n * (g_n - g)(x)| \end{aligned}$$

$$\leq \|f - f_n\|_p \|g_n\|_q$$

$$+ \|f_n\|_p \|g_n - g\|_q$$

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and hence $\|f * g - f_n * g_n\|_{\infty} \rightarrow 0$

which implies $f * g \in C_0(\mathbb{R})$ since

$f_n * g_n \in C_0(\mathbb{R})$. \square