

Appendix to Section 3.

(2) The one point compactification

Also called Alexandrov compactification;
a nice discussion can be found in the
books of Munkres "Topology" § 29.

In fact

Thm. X is a locally compact Hausdorff
space iff there exists a topological space

Y such that:

- (1) X is a top. subspace of Y
- (2) $Y - X$ consists of a single point
- (3) Y is compact Hausdorff.

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If τ, τ' are two spaces satisfying this condition they are homeomorphic by a homeo. restricting to the identity on X .

Given X loc. compact Hausdorff we indicate the construction of such a space;

Let $\beta X := X \cup \{\infty\}$ where $\infty \notin X$.

Topologize βX by defining the collection of open sets of βX to consist of

(1) all sets $U \subset X$ that are open in X

(2) all sets of the form $\beta X \setminus C$

where $C \subset X$ is a compact subset.

One verifies then that this gives a topology

on βX such that (1) βX is compact Hausdorff

(2) $X \subset \beta X$ is a topological subspace; it is open.

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It is dense iff X is not compact.

Let's verify the compactness of βX :

Let $\bigcup_{\alpha \in I} U_\alpha = \beta X$ be a covering of βX

by open subsets. Let $\alpha_0 \in I$ be such that

$U_{\alpha_0} \ni \infty$. Then $U_{\alpha_0} = \beta X \setminus C'$ where

$C' \subset X$ is compact; since $\bigcup_{\alpha \in I} U_\alpha = \beta X$

it follows that $\bigcup_{\alpha \neq \alpha_0} U_\alpha \supset C'$. Since C' is

compact, let $\alpha_1, \dots, \alpha_n$ in I be such that

$\bigcup_{i=1}^n U_{\alpha_i} \supset C'$. Then $\bigcup_{j=0}^n U_{\alpha_j} = \beta X$.

Next we show that

$$C_0(X) = \left\{ f|_X : f \in C(\beta X), f|_{\infty} = 0 \right\}.$$

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Clearly if f is continuous on βX , $f|_X$ is continuous on X ; if $f(\infty) = 0$ then by continuity at ∞ , $\forall \varepsilon > 0$ there exists $U = \beta X \setminus \zeta$ with ζ' compact such that $|f(x)| < \varepsilon \quad \forall x \in U$.

But then $|f|_X(y)| < \varepsilon \quad \forall y \in X \setminus \zeta'$

and hence $f|_X \in C_0(X)$.

Conversely if $g \in C_0(X)$ define

$$f: \beta X \rightarrow \mathbb{C}$$

by $f|_X = g$ and $f(\infty) = 0$. Then

the fact that g vanishes at ~~infinity~~ ^{infinity} \Rightarrow

exactly that f is continuous at ∞ and

$$f(\infty) = 0.$$

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(2) More details on the proof of Thm 3.20.

First A is unital and we want to show \hat{A} is compact.

Consider the map $\Phi: \hat{A} \rightarrow \prod_{a \in A} \overline{D}(0, \|u\|)$

$$\varphi \mapsto (\varphi(a))_{a \in A}.$$

For any given $(\lambda_a)_{a \in A} \in \prod_{a \in A} \overline{D}(0, \|u\|)$

a basis of open neighborhoods of $(\lambda_a)_{a \in A}$ in the product topology is given by

$$\mathcal{V}((\lambda_a); a_1, \dots, a_n; \varepsilon)$$

$$= \left\{ (\lambda'_a) \in \prod_{a \in A} \overline{D}(0, \|u\|) : |\lambda'_{a_i} - \lambda_{a_i}| < \varepsilon, 1 \leq i \leq n \right\}.$$

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It is then a straightforward verification that for

$$\varphi \in \hat{A}:$$

$$\bar{\Phi}(\mathcal{U}(\varphi; a_1, \dots, a_n; \varepsilon)) = \bigvee_{a \in A} \left((\varphi(a)) ; a_1, \dots, a_n ; \varepsilon \right) \cap \bar{\Phi}(\hat{A})$$

and $\mathcal{U}(\varphi; a_1, \dots, a_n; \varepsilon)$

$$= \bar{\Phi}^{-1} \left[\bigvee_{a \in A} \left((\varphi(a)) ; a_1, \dots, a_n ; \varepsilon \right) \cap \bar{\Phi}(\hat{A}) \right]$$

Which shows that $\bar{\Phi}$ is a homeom.

on its image.

Now let $(\lambda_a)_{a \in A}$ be in the closure

of $\bar{\Phi}(\hat{A})$. Let $\varepsilon > 0$, $x, y, x \cdot y, x + y$

be given; then

$$\bigvee_{a \in A} \left((\lambda_a) ; x, y, x \cdot y, x + y ; \varepsilon \right) \cap \bar{\Phi}(\hat{A}) \neq \emptyset.$$

Let $\varphi \in \hat{A}$ such that $\bar{\Phi}(\varphi)$ is in that

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intersection. Then:

$$|\lambda_x - \varphi(x)| < \varepsilon, \quad |\lambda_y - \varphi(y)| < \varepsilon$$

$$|\lambda_{x \cdot y} - \varphi(x \cdot y)| < \varepsilon, \quad |\lambda_{\alpha x + \beta y} - \varphi(\alpha x + \beta y)| < \varepsilon.$$

Thus:

$$\left| \lambda_{\alpha x + \beta y} - \alpha \lambda_x - \beta \lambda_y \right| =$$

$$= \left| \left(\lambda_{\alpha x + \beta y} - \varphi(\alpha x + \beta y) \right) - \alpha \left(\lambda_x - \varphi(x) \right) - \beta \left(\lambda_y - \varphi(y) \right) \right|$$

$$\leq \varepsilon + |\alpha| \varepsilon + |\beta| \varepsilon = \varepsilon (1 + |\alpha| + |\beta|)$$

And:

$$\left| \lambda_{x \cdot y} - \lambda_x \cdot \lambda_y \right| = \left| \lambda_{x \cdot y} - \varphi(x \cdot y) + \varphi(x) \varphi(y) - \lambda_x \cdot \lambda_y \right|$$

$$\leq \varepsilon + \left| \varphi(x) \varphi(y) - \lambda_x \cdot \lambda_y \right|$$

$$= \varepsilon + \left| (\varphi(x) - \lambda_x) \varphi(y) + \lambda_x (\varphi(y) - \lambda_y) \right|$$

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$$\leq \epsilon + \epsilon(|\gamma(y)| + |\lambda_x|) \epsilon$$

$$\leq \epsilon + \epsilon \|y\| + \|x\| \epsilon = \epsilon (1 + \|x\| + \|y\|)$$

And since $\epsilon > 0$ was arbitrary we get

$$\lambda_{\alpha x + \beta y} = \alpha \lambda_x + \beta \lambda_y$$

$$\lambda_{x \cdot y} = \lambda_x \cdot \lambda_y.$$

Define then $\varphi(x) = \lambda_x \quad \forall x \in A$.

Then $\varphi: A \rightarrow \mathbb{C}$ is a \mathbb{C} -algebra homomorphism and since if

$$(\lambda_a)_{a \in A} \in \overline{\Phi(\hat{A})}, \quad \lambda_e = 1 \quad \text{one}$$

has that $\varphi \in \hat{A}$ and $\lambda = \overline{\Phi(e)}$

which shows that $\overline{\Phi(\hat{A})}$ is closed,

hence by Tychonov compact.

(3) Assume now A has no identity and let's consider \hat{A} as subset of \hat{A}_I via the extension $\tilde{\varphi}$ to A_I of a character $\varphi \in \hat{A}$. Then the Gelfand topology on \hat{A} is the one induced by the Gelfand topology on \hat{A}_I .

Indeed let $\varphi \in \hat{A}$ and $\tilde{\varphi} \in \hat{A}_I$.

$$\begin{aligned} \text{Then } \mathcal{U}(\tilde{\varphi}; a_1, \dots, a_n; \varepsilon) \cap \hat{A} \\ = \mathcal{U}(\varphi; a_1, \dots, a_n; \varepsilon) \end{aligned}$$

$$\text{And: } \mathcal{U}(\varphi_\infty; a_1, \dots, a_n; \varepsilon) \cap \hat{A} =$$

$$= \left\{ \varphi \in \hat{A} : |\varphi(a_i)| < \varepsilon \quad 1 \leq i \leq n \right\}.$$

But the latter set is open in \hat{A} : indeed

if $|\varphi(a_i)| < \varepsilon \quad 1 \leq i \leq n$ then

$$\mathcal{U}(\varphi; a_1, \dots, a_n; \varepsilon') \quad \text{for } \varepsilon' = \min \left\{ \frac{\varepsilon - |\varphi(a_i)|}{2} : 1 \leq i \leq n \right\}$$

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is included in $\{ \varphi \in \hat{A} : |P(\varphi, \cdot)| < \varepsilon \text{ (s.t.)} \}$

Thus \hat{A} is the complement of \mathcal{M}_ε in the compact Hausdorff space \hat{A}_I and hence locally compact Hausdorff.

(4) The uniqueness theorem on one point compactifications shows then that \hat{A}_I is homeomorphic to $\beta \hat{A}$ by a homeo. inducing the identity on \hat{A} .