

$\mathcal{L} = \{0\}$ or $\mathcal{L} = \mathcal{H}$.

In this context the following plays a fundamental role:

$$\text{Int}(\pi) = \left\{ T \in \mathcal{L}(\mathcal{X}) : T \overline{\pi(g)} = \overline{\pi(g)} T \forall g \in G \right\}.$$

Then $\text{Int}(\pi)$ is a C^* -subalgebra of $\mathcal{L}(\mathcal{X})$ containing $\{ \lambda I_{\mathcal{X}} : \lambda \in \mathbb{C} \}$.

Observe for instance that if $\mathcal{X} \subset \mathcal{X}$

is a closed invariant subspace and

$\underline{P} : \mathcal{X} \rightarrow \mathcal{L}$ the orthogonal projection,

then $\underline{P} \in \text{Int}(\pi)$.

Thm 5.23 The representation (π, \mathcal{X}) is

irreducible $\iff \text{Int}(\pi) = \{ \lambda I_{\mathcal{X}} : \lambda \in \mathbb{C} \}$

We have

Lemma 5.24 $T \in \mathcal{L}(X)$ normal. Then $T = \lambda \text{Id}_X$

$$\Leftrightarrow |\mathfrak{s}_p(T)| = 1.$$

Proof if $T = \lambda \text{Id}_X$ then $\mathfrak{s}_p(T) = \{\lambda\}$.

Conversely, let E be the resolution of identity on $\mathfrak{s}_p(T)$. Then we know

that $E(\{\lambda\}) = E(\mathfrak{s}_p(T)) = \text{Id}_X$ but

also, that since $\{\lambda\}$ is an isolated point $E(\{\lambda\})$ is the projection to

$\text{Ker}(T - \lambda \text{Id})$, hence $T = \lambda \text{Id}$.

□

Proof of Thm.

(1) We have already observed that if

$\mathcal{O} \subset \mathcal{L} \subset \mathcal{H}$ closed invariant, then

$$\text{Int}(\mathcal{O}) \supseteq \{\lambda \text{Id} : \lambda \in \mathbb{C}\}.$$

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(2) Assume $\text{Int}(\pi) \supseteq \{\lambda I_d : \lambda \in \mathbb{C}\}$

then $\{T \in \text{Int}^+(\pi) : T = T^*\} \supsetneq R \cdot \text{Id}_{\mathcal{H}}$

Pick a $T = T^* \in \text{Int}^+(\pi)$, not in

$R \cdot \text{Id}_{\mathcal{H}}$. It follows from the lemma

that $|s_p(T)| \geq 2$ and hence there

are $V_i \subset s_p(T)$ open, non-empty and

$V_1 \cap V_2 = \emptyset$. Hence $E(V_1) \neq 0$, $E(V_2) \neq 0$

and since $\pi^{(g)}$ commutes with T

$\forall g \in G$, it commutes with $E(V_1)$ and

$E(V_2)$. Clearly $\mathcal{Z} := \overline{\text{Im } E(V_1)}$ is

a closed invariant subspace with

$0 \subseteq \mathcal{Z} \subseteq \mathcal{H}$. \blacksquare

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Positive Operators and Polar

Decomposition.

We first study a special case of self-adjoint operators, namely:

Definition 5.24 We say that $T \in \mathcal{L}(\mathcal{H})$

is positive if $\langle Tx, x \rangle \geq 0 \quad \forall x \in \mathcal{H}$.

In this case we write $T \geq 0$.

Observe that it is implicit in the definition

that $\langle Tz, z \rangle \in \mathbb{R} \quad \forall z \in \mathcal{H}$ which

implies $\langle x, Tx \rangle = \langle Tz, x \rangle$ and T

is self-adjoint.

Thm 5.25 Let $T \in \mathcal{L}(\mathcal{H})$. TFAE:

(1) T is positive

(2) $T^* = T$ and $\text{Sp}(T) \subset [0, \infty[$.

Proof: (1) \Rightarrow (2): we already observed that

$T^* = T$; this implies $\text{Sp}(T) \subset \mathbb{R}$. Let

$\lambda > 0$, then

$$\lambda \|x\|^2 = \langle \lambda x, x \rangle \leq \langle (T + \lambda I)x, x \rangle$$

$$\leq \|T + \lambda I\| \|x\|^2$$

~~that is~~ $\lambda \leq \|T + \lambda I\|$, hence

thus $\|(T + \lambda I)x\| \geq \lambda \|x\| \quad \forall x \in \mathcal{H}$.

Since T is self-adjoint it is normal,

so is $T + \lambda I$ and hence $T + \lambda I$ is

invertible by Prop 5.9 (3) which implies

$-\lambda \notin \text{Sp}(T)$ and shows that $\text{Sp}(T) \subset [0, \infty[$.

(2) \Rightarrow (1) This uses the spectral theorem.

Let E be the resolution of identity on

$\text{Sp}(T)$ so that

$$\langle T_{\infty}, \chi \rangle = \int_{\sigma_p(T)} \lambda dE_{\infty}(\lambda)$$

Since E_{∞} is a positive measure and

$\lambda \geq 0$ on $\sigma_p(T)$, we have $\langle T_{\infty}, \chi \rangle \geq 0$

$\forall \chi \in \mathcal{H}$.



The following theorem is about a single positive operator; nevertheless it uses the Gelfand isomorphism for abelian C^* -algebras.

Thm 5.26 Every positive $T \in \mathcal{L}(\mathcal{H})$

has a unique positive square root S .

If T is invertible so is S .

Proof: Let E be the resolution of

identity with $T = \int \lambda dE(\lambda)$.

$\sigma_p(T)$

Since $\sigma_p(T) \subset [0, +\infty[$ we can define

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$$\zeta := \int \sqrt{\lambda} \, dE(\lambda)$$
$$S_p(T)$$

using just the continuous Functional calculus.

Then $\zeta^2 = T$, $\zeta = \zeta^*$ and

$S_p(\zeta) \subset [0, \infty]$, in fact $S_p(\zeta) = S_p(T)$

Let then $\zeta' = (\zeta')^*$ with $S_p(\zeta') \subset [0, \infty]$

and $(\zeta')^2 = T$. Let E' be

the resolution of Id for ζ' :

$$\zeta' = \int \lambda \, dE'(\lambda)$$
$$S_p(\zeta')$$

Then $T = \zeta'^2 = \int \lambda^2 \, dE'(\lambda)$

hence $\forall f \in C([0, \infty])$:

$$f(T) = f(\zeta'^2) = \int f(\lambda^2) \, dE'(\lambda)$$
$$S_p(\zeta')$$
$$= \int f(\lambda) \, dE(\lambda)$$
$$S_p(T)$$

Hence $\forall x \in \mathcal{H}$:

$$\int f(\lambda) dE_{x,x}(\lambda) = \int f(\lambda^2) dE'_{x,x}(\lambda)$$

which by replace f by $f(\lambda) = g(\sqrt{\lambda})$

gives $\int g(\lambda) dE'_{x,x}(\lambda) = \int g(\sqrt{\lambda}) dE_{x,x}(\lambda)$

and implies that $E'_{x,x}$ is uniquely determined by $E_{x,x}$, hence E' is uniquely determined by E which implies uniqueness.

Finally if T is invertible: $S^* = T^{-1}S$

since S commutes with $S^* = T$. \square

Now if $T \in \mathcal{L}(\mathcal{H})$ is arbitrary then

observe that $\langle T^*T x, x \rangle = \|Tx\|^2 \geq 0$

and hence $T^*T \geq 0$.

This leads to

Thm 5.27: If $T \in L(\mathcal{H})$ the positive square root of T^*T is the only positive operator $P \in L(\mathcal{H})$ satisfying

$$\|Px\| = \|Tx\| \quad \forall x \in \mathcal{H}.$$

Proof: If P is the positive square root
 $\sqrt{T^*T}$ we have

$$\|Px\|^2 = \langle P^2x, x \rangle = \langle T^*Tx, x \rangle = \|Tx\|^2$$

$\forall x.$

Conversely if P' is a positive operator with

$$\|P'x\| = \|Tx\|$$

then the computation above shows that

$$\langle P'^2x, x \rangle = \langle T^*Tx, x \rangle$$

hence $P'^2 = T^*T$ and hence $P' = P$

by the uniqueness statement in Thm 5.26. ③

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This allows us to generalize to Hilbert Spaces a classical fact from linear algebra, namely that every invertible matrix $T \in GL(n, \mathbb{C})$ has a unique decomposition $\overline{T} = U \cdot P$ where $U \in U(n)$ is unitary and P is hermitian, that is ${}^t \bar{P} = P$ and positive definite.

Thm 5.28

Let $T \in \mathcal{L}(X)$ be invertible. Then

$$T = U \cdot P$$

with U unitary and $P \geq 0$ positive.

Moreover this decomposition is unique.

Proof : Since T is invertible, \tilde{T}^* is and so is $\tilde{T}T\tilde{T}$; Thm 5.2c implies that the positive square root P of $\tilde{T}T\tilde{T}$ is invertible as well. Let $U := \tilde{T}P^{-1}$.

Then

$$*U^*U = *P^{-1}\tilde{T}T\tilde{T}P^{-1} = \tilde{P}^*P^2\tilde{P}^{-1} = \text{Id}.$$

Observe that U is invertible as well and hence unitary.

Now if $T = U'P'$ with U' unitary and $P' \geq 0$, then:

$$\begin{aligned} *T^*T &= (P')^* (U')^* U' P' \\ &= \tilde{P}'^* P' = P'^2 \end{aligned}$$

By the uniqueness statement in Thm 5.2c we get $P' = P$ and hence $U' = U$.