

extend those properties of self-adjoint elements to general  $C^*$ -algebras.

Prop 4.4. Let  $A$  be a unital  $C^*$ -algebra.

(1) If  $u$  is unitary,  $\text{Sp}_A(u) \subset \mathbb{T}$

(2) If  $h$  is self-adjoint,  $\text{Sp}_A(h) \subset \mathbb{R}$ .

Using Prop. 4.3 we obtain:

Corollary 4.5 Let  $A$  be a  $C^*$ -algebra and  $h \in A$  self-adjoint. Then  $\text{Sp}_A(h) \subset \mathbb{R}$ .

Proof of Prop. 4.4.

(1) First we show that if  $f: \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic and  $x \in A$  then

$$f(\text{Sp}_A(x)) \subset \text{Sp}_A(f(x)).$$

This is a very special case of the

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holomorphic functional calculus (see Thm 2.10)

and is elementary: indeed let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

the Taylor series at  $\circ$  of  $f$  which by hypothesis converges absolutely in  $A$ ;  
hence so does

$$f(x) := \sum_{n=0}^{\infty} a_n x^n, \quad x \in A,$$

where we set  $x^0 = e$ . Then:

$$f(x) - f(\lambda) \cdot e = \sum_{n=0}^{\infty} a_n (x^n - \lambda^n e)$$

Now factor

$$x^n - \lambda^n e = (x - \lambda e) \underbrace{(e + x^{n-1} \lambda + \dots + \lambda^{n-1} e)}_{p_n(x, \lambda)}$$

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$$\text{Now } \|P_n(x, \lambda)\| \leq ((\|x\|^{n-1} + \|\lambda e\|^{n-1})\|x\| + \dots + \lambda^{n-1})$$
$$\leq n \cdot r^{n-1}$$

where  $r = \max(\|x\|, \|\lambda e\|)$ .

Now  $\sum_{n=0}^{\infty} a_n n z^{n-1}$  is the Taylor series of the derivative of  $f$ , hence converges absolutely in  $\mathbb{C}$ ; thus

$$\sum_{n=0}^{\infty} |a_n| n r^{n-1}$$

converges and so does

$$p(x, \lambda) := \sum_{n=0}^{\infty} P_n(x, \lambda) \in A.$$

$$\text{So } f(x) - f(\lambda)e = (x - \lambda e)p(x, \lambda)$$

and hence if  $\lambda \in \text{Sp}_A(x)$ , it follows that  $f(\lambda) \in \text{Sp}_x(f(x))$ .

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(2) Observe first that in a unital C\*-algebra we have  $e = e^*$  since both  $e$  and  $e^*$  are units, and  $\|e\|^2 = \|ee^*\| = \|e^2\| = \|e\|$  hence  $\|e\| = 1$ .

If now  $u \in A$  is unitary:

$$\|u\|^2 = \|uu^*\| = \|e\| = 1 \text{ hence}$$

$$\|u\| = 1.$$

Let  $\lambda \in \mathbb{C}$  with  $|\lambda| < 1$ . Then

$$u - \lambda e = (e - \lambda u^*)u, \text{ and since}$$

$\|\lambda u^*\| = |\lambda| < 1$ ,  $e - \lambda u^*$  is invertible

and so is  $u - \lambda e$ .

$$\text{If } |\lambda| > 1, u - \lambda e = \lambda \left( \frac{u}{\lambda} - e \right)$$

and we reach the same conclusion using

$$\left\| \frac{u}{\lambda} \right\| = \frac{1}{|\lambda|} < 1.$$

Hence  $\text{Sp}_A(h) \subset \overline{\mathbb{H}}$ .

(3) If  $h = h^*$  then  $\exp(i \cdot h)$  is unitary and hence by (1):

$$\exp i \text{Sp}_A(h) \subset \text{Sp}_A(\exp ih) \subset \overline{\mathbb{H}}$$

which implies  $\text{Sp}_A(h) \subset \mathbb{R}$ .

□

Theorem 4.6. Let  $A$  be an abelian  $C^*$ -algebra

and  $\hat{A}$  its Gelfand spectrum. Then

the Gelfand transform

$$\hat{\cdot}: A \rightarrow C_0(\hat{A})$$

is a  $*$ -isomorphism.

The proof will rely on a classical approximation result, namely:

Stone-Wijskars: Let  $X$  be l.c. Hausdorff.

Let  $\mathcal{A} \subset C_0(X)$  be a subalgebra

such that: (1)  $f \in \mathcal{A} \rightarrow \bar{f} \in \mathcal{A}$

(2)  $\forall x \in X \exists f \in \mathcal{A}, f(x) \neq 0$ .

(3)  $\forall x \neq y \exists f \in \mathcal{A}, f(x) \neq f(y)$

Then  $\mathcal{A}$  is dense in  $C_0(X)$  for  $\|\cdot\|_\infty$ .

### Proof of Thm 4.6

Since  $\mathcal{A}$  is abelian, every element is normal and hence Thm 3.23 and Prop. 4.2 imply that  $\forall a \in \mathcal{A}$ :

$$\|\hat{a}\|_\infty = \|a\|.$$

Let  $B = \{\hat{a} : a \in \mathcal{A}\} \subset C_0(\hat{A})$ .

Then, since  $\mathcal{A}$  is complete and  $a \mapsto \hat{a}$  is isometric,  $B$  is complete

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and hence closed in  $C_0(\hat{A})$ .

Next we want to show that  $\mathcal{B}$  satisfies the assumption of the Stone Weierstrass theorem.

(1) Let  $a = a_0 + i\alpha_1$ , with  $a_0, \alpha_1$  hamiltonian. Then  $a^* = a_0 - i\alpha_1$  and hence  $\widehat{a^*}(x) = \chi(a_0) - i\chi(\alpha_1)$ .

Put  $\chi(a_0) \in \overline{\text{Sp}_1(\alpha_0)} \subset \mathbb{R}$  by Corollary 4.5 and for the same reason  $\chi(\alpha_1) \in \mathbb{R}$ . Thus:

$$\widehat{a^*}(x) = \overline{\widehat{a}(x)}$$

which implies in particular that if  $f \in \mathcal{B}$  then  $\bar{f} \in \mathcal{B}$ .

(2) By the definition of character,

$\forall x \in \hat{A}$  there is  $a \in A$  with  $x(a) \neq 0$   
that is  $\hat{x}(x) \neq 0$ .

(3) If  $x_1 \neq x_2$  then by definition  
there is  $a \in A$  with  $x_1(a) \neq x_2(a)$   
that is  $\hat{x}_1(x_1) \neq \hat{x}_2(x_2)$ .

It follows from Stone-Weierstrass, that  
 $B$  is dense in  $C_0(\hat{A})$  and being  
closed we get  $B = C_0(\hat{A})$ . This concludes  
the proof of the theorem.  $\square$

It is natural to enquire about the  
functorial nature of the map that to  
an abelian  $C^*$ -algebra  $A$  associates  $C_0(\hat{A})$ .

Corollary 4.7 For two abelian  $C^*$ -algebras

$A, B$  the following are equivalent:

(1)  $A$  and  $B$  are isomorphic as  $C$ -algebras.

(2)  $\hat{A}$  and  $\hat{B}$  are homeomorphic

(3)  $A$  and  $B$  are isometrically  $*-\tilde{\pi}_0$ -isomorphic.

Proof:

The proof will show a more precise statement. Let  $T: A \rightarrow B$  be a  $C$ -algebra isomorphism. Since  $B$  is semisimple,  $T$  is continuous (Cor. 3.16) and since  $A$  is semisimple,  $T^{-1}$  is continuous.

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From this follows that  $x \in \hat{B}$  implies  
 $x \circ T \in \hat{A}$  and the map

$$t: \hat{B} \rightarrow \hat{A}$$

$$x \mapsto x \circ T$$

is a bijection. Let  $x_0 \in \hat{A}$ ,  $a_1, \dots, a_n$  in  
 $A$  and  $\varepsilon > 0$ : then if  $|x(a_i) - x_0(a_i)| < \varepsilon$   
we have  $|(\bar{x} \circ \bar{T})(T(a_i)) - x_0 \circ \bar{T}(T(a_i))| < \varepsilon$

which shows that

$$\bar{t}^{-1}(u(x_0; a_1, \dots, a_n; \varepsilon)) = u\left(\bar{t}'(x_0), T(a_1), \dots, T(a_n); \varepsilon\right)$$

and shows that  $t$  is a homeomorphism.

Thus the map  $\delta: C_0(\hat{A}) \rightarrow C_0(\hat{B})$

$$f \mapsto f \circ t$$

is an isometric  $\star$ -isomorphism.

This shows  $(1) \Rightarrow (2) \Rightarrow (3)$ ; while  
 $(3) \Rightarrow (1)$  is obvious. Observe that

the diagramme

$$\begin{array}{ccc} A & \xrightarrow{T} & B \\ \downarrow & & \downarrow \\ C_0(\hat{A}) & \xrightarrow{f} & C_0(\hat{B}) \end{array}$$

commutes which implies that  $T$  is an isometric \*-isomorphism.  $\square$

As next application we will prove an "abstract spectral theorem" for normal operators. First we have to deal with the following issue: if  $A$  is a Banach algebra with identity  $e$  and  $B$  a subbanach algebra with  $B \ni e$  then clearly if for  $x \in B$  and  $\lambda \in \mathbb{C}$