

2. The Spectrum of a Banach Algebra

Element.

Let A be a \mathbb{C} -algebra with unit $e \in A$.

Def 2.1 : An element $x \in A$ is called invertible if there exists $y \in A$ with $xy = yx = e$. Then y is unique and denoted x^{-1} .

The set $G(A)$ of invertible elements of A is a group with ~~multiplication~~ neutral element e .

For $x \in A$ define :

$$\text{Sp}_A(x) := \left\{ \lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible} \right\}.$$

It is called the spectrum of x and its

complement $P_A(x) := \mathbb{C} - \text{Sp}_A(x)$

the resolvent set of x .

When A does not have an identity

we define

$$\text{Sp}_A(x) = \text{Sp}_{A_I}(x)$$

$$\text{and } f_A(x) = P_{A_I}(x).$$

Example 2.2 Let $A = \text{End}(V)$ where

V is a finite dimensional \mathbb{C} -vector space. Then :

$\text{Sp}_A(T) = \text{set of roots of the characteristic polynomial of } T$. This follows

from: ~~$T - \lambda \text{Id}$~~ not invertible iff

$\text{Ker}(T - \lambda \text{Id}) \neq \{0\}$, which uses $\dim V < \infty$.

The fact that $\text{Sp}_A(T) \neq \emptyset$ follows from

the fundamental theorem of algebra.

One of the main results of this section is that for a general Banach algebra the spectrum of an element is never empty.

In fact we will prove a formula, the spectral radius formula, for the radius of the smallest closed disc centered at $0 \in \mathbb{C}$ and containing $\text{Sp}_A(x)$. This formula will involve the following invariant of an element $x \in A$:

$$\underline{\text{Def 2.3}} \quad r_A(x) = \inf \left\{ \|x^n\|^{1/n} : n \geq 1 \right\}.$$

Since $\|x^n\| \leq \|x\|^n$ we obtain

$$r_A(x) \leq \|x\|.$$

In addition we have

$$r_A(\lambda x) = |\lambda| r_A(x) \quad \forall \lambda \in \mathbb{C}, \\ \forall x \in A.$$

The fact that this infimum is a limit turns out to be essential in the sequel.

It will follow from the following lemma applied to $f(n) = \|x^n\|$:

Lemma 2.4 Let $f: \mathbb{N}^* \rightarrow \mathbb{R}_{\geq 0}$ with

$$f(n+m) \leq f(n) \cdot f(m), \quad \forall n, m \in \mathbb{N}^*.$$

Then $\lim_{n \rightarrow \infty} f(n)^{\frac{1}{n}}$ exists and equals

the infimum of $\{f(n)^{\frac{1}{n}} : n \geq 1\}$.

Proof: Since $f(n) \geq 0 \quad \forall n \geq 1$ the infimum $r := \inf \{f(n)^{\frac{1}{n}} : n \geq 1\}$ exists.

Let $\varepsilon > 0$ and $k \geq 1$ with $f(k)^{\frac{1}{k}} < r + \varepsilon$.

Let $n \geq k$ and $n = a \cdot k + b$ with

$a \geq 1$ and $0 \leq b \leq k-1$. Then:

$$f(n)^{\frac{1}{n}} = \|f(a \cdot k + b)\|^{\frac{1}{n}} \leq \|f(k)\|^{\frac{a}{n}} \|f(1)\|^{\frac{b}{n}}$$

$$f(n)^{\frac{1}{n}} \leq \|f(k)\|^{\frac{a}{n}} \|f(1)\|^{\frac{b}{n}}.$$

$$\text{Now } 1 = \frac{a}{n} k + \frac{b}{n}$$

$$\text{that is: } \frac{a}{n} = (1 - \frac{b}{n}) \frac{1}{k}.$$

$$\begin{aligned} \text{Thus: } & \lim_{n \rightarrow \infty} (1 - \frac{b}{n}) \frac{1}{k} & b/n \\ f(n) & \leq \varphi(n) & f(1) \\ & \leq (r + \varepsilon) & f(1) \\ & & b/n \end{aligned}$$

$$\text{Thus } \limsup_{n \rightarrow \infty} f(n) \leq r + \varepsilon$$

$\forall \varepsilon > 0$ which implies:

$$\limsup_{n \rightarrow \infty} f(n) \leq r \leq \liminf_{n \rightarrow \infty} f(n)$$

which concludes the proof. \blacksquare

$$\underline{\text{Corollary 2.5}} \quad \Gamma(x) = \lim_{n \rightarrow \infty} \|x^n\|$$

Proof: Indeed, $\|x^{n+m}\| \leq \|x^n\| \cdot \|x^m\|$.

Now apply Lemma 2.4 with $f(n) = \|x^n\|$.

\blacksquare

In the next few lemmas we give criteria for an element to belong to $G(A)$ and deduce topological properties for $G(A)$.

Lemma 2.6. Let A be a Banach algebra with identity $e \in A$ and $x \in A$ with $r_A(x) < 1$. Then $e-x$ is invertible

and $(e-x)^{-1} = e + \sum_{n=1}^{\infty} x^n$

where the series is absolutely convergent.

Terminology: If A is a Banach space

we say that $\sum_{n=1}^{\infty} a_n$, $a_n \in A$

converges if the sequence of partial sums $s_n := \sum_{n=1}^{\infty} a_n$ converges in A.

We say that it converges absolutely if

$$\sum_{n=1}^{\infty} \|a_n\|$$

converges. It is an easy exercise that absolute convergence implies convergence.

Proof: Pick $\frac{r}{A} < \gamma < 1$ and by

~~Lemma~~ Corollary 2.5 let $N \geq 1$

such that $\|x^n\|^{\gamma^n} \leq \gamma \quad \forall n \geq N$

that is $\|x^n\| \leq \gamma^n \quad \forall n \geq N$.

But this implies that $\sum_{n=1}^{\infty} \|x^n\|$

converges hence $\sum_{n=1}^{\infty} x^n$ converges in A.

- 2 - 8 -

Let $s_n = e + \sum_{k=1}^n x^k$

and $y := \lim_{n \rightarrow \infty} s_n$.

Then: $s_n \cdot x = x + \sum_{k=1}^n x^{k+1}$

$$= s_{n+1} - e$$

and $x \cdot s_n = s_{n+1} - e$

which by taking limits implies

$$y \cdot x = y - e$$

$$x \cdot y = y - e$$

and hence $e = y \cdot (e - x) = (e - x) \cdot y$

□

Finally we will need that in the group $G(A)$ the inverse $x \mapsto x^{-1}$ is a continuous map, and that $G(A)$ is open in A . This follows from

Lemma 2.7 Let A be a Banach algebra with identity $e \in A$.

(1) If $x, y \in G(A)$ with $\|y-x\| \leq \frac{1}{2}\|x^{-1}\|^{-1}$ then $\|y^{-1}-x^{-1}\| \leq 2\|x^{-1}\|^2 \|y-x\|$.

In particular the map $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$.

(2) $G(A)$ is open in A and if $x \in A$ is such that $\|x-e\| < 1$ then $x \in G(A)$.

Proof:

(1) The idea is to write :

$$\bar{y}' - \bar{x}' = \bar{y}'(x - y)x^{-1}$$

thus $\|\bar{y}' - \bar{x}'\| \leq \|\bar{y}'\| \|x - y\| \|\bar{x}'\| \quad (*)$

We want to bound $\|\bar{y}'\|$: for this

We use (*) in the following way :

$$\|\bar{y}'\| - \|\bar{x}'\| \leq \|\bar{y}' - \bar{x}'\| \leq \|\bar{y}'\| \|x - y\| \|\bar{x}'\|$$

which together with the hypothesis

$$\|x - y\| \|\bar{x}'\| \leq \frac{1}{2}$$

gives $\|\bar{y}'\| - \|\bar{x}'\| \leq \frac{1}{2} \|\bar{y}'\|$

and thus $\|\bar{y}'\| \leq 2 \|\bar{x}'\|$.

Inserted into (*) proves (1).

- 2-4 - .

(2) If $\|e-x\| < 1$ then $\zeta_A(e-x) < 1$
and $e-(e-x) = x \in G(A)$.

Now let $x \in G(A)$ and $y \in A$
with $\|y-x\| < \|x'\|^{-1}$. Then

$$\begin{aligned}\|x'y-e\| &= \|x'(y-x)\| \\ &\leq \|x'\|\|y-x\| < 1\end{aligned}$$

which implies $\zeta_A'y \in G(A)$ and
hence $y \in G(A)$. ■