

5.2. Operators in Hilbert Spaces.

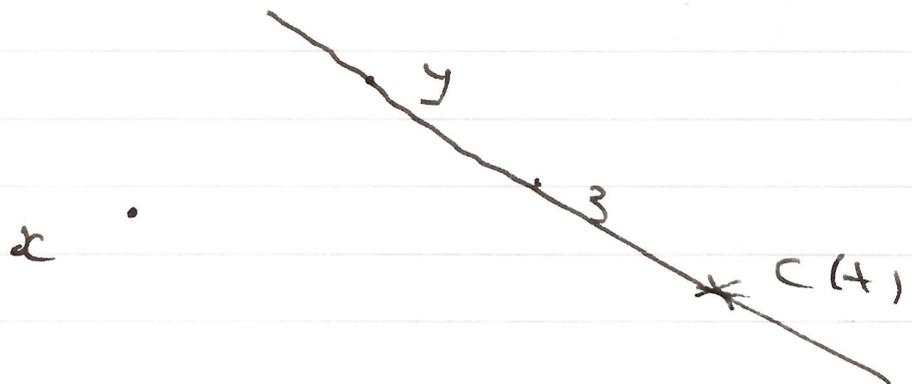
Let \mathcal{H} be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. We will denote by $d(x, y) := \|x - y\|$ the distance between x and y .

Let $x, y, z \in \mathcal{H}$ and consider

$$c: \mathbb{R} \longrightarrow \mathcal{H}$$

$$t \longmapsto ty + (1-t)z$$

the straight line through y and z .



A fundamental geometric fact is that

when $y \neq z$ the function :

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$$f(t) := d(x, c(t))^2 = \|x - (ty + (1-t)z)\|^2$$

is strictly convex; indeed a computation

gives $f''(t) = 2\|z-y\|^2$.

With this one shows:

Thm 5.4 Let $C \subset \mathcal{H}$ be closed

convex and $x \in \mathcal{H}$. Then there exists

a unique point $y \in C$ with

$$d(x, y) = \inf \{ d(x, z) : z \in C \}.$$

If $A \subset \mathcal{H}$ is any subset, the ortho-

gonal A^\perp of A ,

$$A^\perp = \{ x \in \mathcal{H} : \langle x, a \rangle = 0 \quad \forall a \in A \}$$

is a closed vector subspace of \mathcal{H} ,

and we have $\overline{A}^\perp = A^\perp$.

From this and Thm 5.4 one deduces

Thm 5.5 Let $E \subset \mathcal{H}$ be a vector subspace then we have an orthogonal direct sum decomposition:

$$\overline{E} \oplus E^\perp = \mathcal{H}.$$

Proof: Since $E^\perp = \overline{E}^\perp$ we may assume

E is closed. Let $x \in \mathcal{H}$ and $y \in E$

with $d(x, y) = \min \{ d(x, z) : z \in E \}$

(Thm 5.4). For every $m \in E$, the

smooth function $d(t) := d(x, y + tm)^2$

has a minimum at $t = 0$. Hence

$d'(0) = 0$; but $d'(0) = -\langle m, x - y \rangle$

which shows $x - y \in E^\perp$. \square

Corollary 5.6 If $E \subset \mathcal{H}$ is a vector subspace then $\overline{E} = (E^\perp)^\perp$.

Now we turn to some basic facts concerning bounded operators in \mathcal{H} and in particular normal operators.

Prop. 5.7 For $T \in \mathcal{L}(\mathcal{H})$, $x, y \in \mathcal{H}$ we have:

$$2\langle Tx, y \rangle = \langle T(x+y), (x+y) \rangle + i \langle T(x+iy), (x+iy) \rangle \\ - (1+i) \langle T(x), x \rangle - (1-i) \langle T(y), y \rangle.$$

In particular for $T, S \in \mathcal{L}(\mathcal{H})$ we have

$$\langle Tx, x \rangle = \langle Sx, x \rangle \quad \forall x \in \mathcal{H}$$

$$\Leftrightarrow T = S.$$

Recall that the adjoint T^* of a bounded operator $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\langle T^*x, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in \mathcal{H}.$$

This defines an involution in $\mathcal{L}(\mathcal{H})$ and we have shown that $\mathcal{L}(\mathcal{H})$ is a C^* -algebra.

Prop. 5.8 If $T \in \mathcal{L}(\mathcal{H})$ then

$$\text{Ker}(T^*) = \text{Im}(T)^\perp \quad \text{and} \quad \text{Ker}(T) = \text{Im}(T^*)^\perp$$

Proof: The second follows from the first since $T^{**} = T$. For the first:

$$T^*y = 0 \iff \langle x, T^*y \rangle = 0 \quad \forall x \in \mathcal{H}$$

$$\iff \langle Tx, y \rangle = 0 \quad \forall x \in \mathcal{H}$$

$$\iff y \in \text{Im}(T)^\perp \quad \square$$

Now we establish some basic facts concerning normal operators:

Prop. 5.9 An operator $T \in \mathcal{L}(\mathcal{H})$ is normal iff $\|Tx\| = \|T^*x\| \quad \forall x \in \mathcal{H}$.

In addition a normal operator has the following properties:

(1) $\text{Ker } T = \text{Ker } T^*$

(2) $\overline{\text{Im}(T)} = \mathcal{H} \iff \text{Ker } T = 0$

(3) T is invertible $\iff \exists c > 0$ with $\|Tx\| \geq c\|x\|, \forall x \in \mathcal{H}$.

(4) If $Tx = \alpha \cdot x$ for some $x \in \mathcal{H}$ then $T^*x = \bar{\alpha} \cdot x$

(5) If $\alpha \neq \beta$ are eigenvalues of T then the corresponding eigenspaces are orthogonal.

Proof: We have $\forall x \in \mathcal{H}$:

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle$$

$$\|T^*x\|^2 = \langle T^*x, T^*x \rangle = \langle TT^*x, x \rangle$$

this combined with Prop. 5.7 proves the first assertion, from which $\text{Ker}(T) = \text{Ker}(T^*)$ follows.

Next (Prop. 5.8) we have $\text{Ker}(T^*) = \text{Im}(T)^\perp$ hence $\text{Ker } T = \text{Im}(T)^\perp$;

thus $\text{Ker } T = 0 \Leftrightarrow \text{Im}(T)^\perp = 0$ and

since $\text{Im}(T)^\perp{}^\perp = \overline{\text{Im}(T)}$ the last

is equivalent to $\overline{\text{Im}(T)} = \mathcal{H}$.

Next : if $\|Tx\| \geq c\|x\| \quad \forall x \in \mathcal{H}$

for a constant $c > 0$, it follows that

$\text{Ker } T = 0$ and $\text{Im}(T)$ is closed which

in view of (2) implies $\text{Im}(T) = \mathcal{H}$

and T is invertible. The converse follows from the open mapping theorem.

Concerning ψ : $(T - \alpha \text{Id})^* = T^* - \bar{\alpha} \text{Id}$

hence by (1): $\text{Ker}(T - \alpha \text{Id}) = \text{Ker}(T^* - \bar{\alpha} \text{Id})$.

Finally if $Tx = \alpha x$, $Ty = \beta y$ we

have

$$\begin{aligned} \alpha \langle x, y \rangle &= \langle \alpha x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle \\ &= \langle x, \bar{\beta} y \rangle = \bar{\beta} \langle x, y \rangle \end{aligned}$$

with $\alpha \neq \bar{\beta}$ implies $\langle x, y \rangle = 0$. \square

Finally we discuss a few characterizations of self-adjoint projections.

Recall that $P \in \mathcal{L}(X)$ is called a projection if $P^2 = P$.

Prop. 5.10 Let $P \in \mathcal{L}(\mathcal{H})$ be a projection.

TFAE:

(1) P is self-adjoint

(2) P is normal

(3) $\text{Im}(P) = \text{Ker}(P)^\perp$

(4) $\langle P(x), x \rangle = \|P(x)\|^2 \quad \forall x \in \mathcal{H}$.

~~Proof~~: Moreover for two self-adjoint projections P, Q we have

$$\text{Im}(P) \perp \text{Im}(Q) \iff P \cdot Q = 0.$$

Proof:

(1) \iff (2) clear.

Since P is normal, Prop. 5.9 (1) and 5.8 imply $\text{Ker}(P) = \text{Im}(P)^\perp$.

Since P is a projection we have $x = Py$ for some $y \in \mathcal{H} \iff Px = x$ which

shows that $\text{Im}(P) = \text{Ker}(I - P)$

and in particular $\text{Im}(P)$ is closed.

Hence $\text{Im}(P) = \text{Im}(P)^{\perp\perp} = \text{Ker}(P)^{\perp}$.

Assume $\text{Im}(P) = \text{Ker}(P)^{\perp}$:

then $\mathcal{H} = \text{Im}(P) \oplus \text{Ker}(P)$; let

$x \in \mathcal{H}$, $x = y + z$ with $\langle y, z \rangle = 0$

$Py = y$ and $Pz = 0$. Then $Px = y$ and

$$\langle Px, x \rangle = \langle y, y + z \rangle = \langle y, y \rangle = \|Px\|^2.$$

Assume now $\langle Px, x \rangle = \|Px\|^2 \quad \forall x \in \mathcal{H}$.

$$\begin{aligned} \text{Then } \|Px\|^2 &= \langle Px, x \rangle = \langle x, P^*x \rangle \\ &= \langle P^*x, x \rangle \end{aligned} \quad \left(\|Px\|^2 \text{ is real} \right)$$

which implies $P = P^*$.

$$\text{Finally: } \langle Px, Qy \rangle = \langle x, P^*Qy \rangle$$

which proves the last assertion. \square

5.3. An Example.

Let T be a normal operator in $\mathcal{L}(H)$; as in Thm 4.10 we consider the abelian sub C^* -algebra generated by T :

$$B := \overline{\{P(T, T^*) : P \in \mathbb{C}[x, \bar{x}]\}}$$

Let $Sp(T) \subset \mathbb{C}$ be the spectrum of T (always seen as element of $\mathcal{L}(H)$).

Then Thm 4.10 provides us with a C^* -algebra isomorphism

$$C(Sp(T)) \longrightarrow B, \quad f \mapsto f(T)$$

sending $\mathbb{1}$ to Id_H and id to T .

Lemma 5.11 Assume $\{\lambda_0\}$ is an isolated point of $Sp(T)$, then λ_0 is an eigenvalue of T .

Proof: the characteristic function

χ_{λ_0} of $\{\lambda_0\}$ is continuous

on $\text{Sp}(T)$. Let $P := \chi_{\lambda_0}(T)$:

then it follows from $\chi_{\lambda_0} \cdot \chi_{\lambda_0} = \chi_{\lambda_0}$

and $\overline{\chi_{\lambda_0}} = \chi_{\lambda_0}$ that P is a self-

-adjoint projection. Now we have

$$(id - \lambda_0 Id) \chi_{\lambda_0} = 0$$

which implies $(T - \lambda_0 Id) P = 0$.

Since $\chi_{\lambda_0} \neq 0$, $\text{Im } P \neq \{0\}$ and

λ_0 is an eigenvalue of T . \square

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Take now $\mathcal{H} = \ell^2(\mathbb{Z})$ and $Tf(x) := f(x+1)$. Then $T^y f(x) = f(x-y)$

and T is unitary; hence $\sigma_p(T) \subset \mathbb{T}$.

Then $\sigma_p(T)$ is a compact, not-empty subset of \mathbb{T} without isolated

points. Indeed if $\lambda_0 \in \sigma_p(T)$

where an isolated point, it would be an eigenvalue. Let then $f \in \ell^2(\mathbb{Z})$

with $f(x+1) = Tf(x) = \lambda_0 f(x)$

$\forall x \in \mathbb{Z}$.

But this implies $f(x) = \lambda_0^x f(0)$

and $\sum_{x \in \mathbb{Z}} |f(x)|^2 = \sum_{x \in \mathbb{Z}} |\lambda_0|^x |f(0)|^2$

$\Leftarrow < +\infty$

$\Leftrightarrow f(0) = 0$ which implies $f = 0$.