

## 6.2. Some miraculous facts about topological groups.

Recall that a topological space is connected if it cannot be written as disjoint union of open non-empty subsets. Recall that the closure of a connected set is connected and the continuous image of a connected set is connected. Finally given a topological space  $X$ , the relation  $x \sim y$  if  $\{x, y\}$  is contained in a connected subset of  $X$  is an equivalence relation and its equivalence classes are called connected components.

Prop. 6.4. Let  $G$  be a topological group.

Then we have

(1) If  $H \leq G$  is a subgroup, so is its closure  $\overline{H}$ .

(2) If  $H < G$  is an open subgroup then it is closed.

(3) The connected component  $G_0$  of  $G$  containing the identity is a closed normal subgroup.

(4) If  $G$  is connected and  $V \ni e$  is a neighborhood of  $e$  then  $V \cup V^{-1}$  generates  $G$ , that is:

$$G = \bigcup_{n \geq 1} (V \cup V^{-1})^n$$

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Notation give subset  $A, B \subset G$  we denote  $A \cdot B := \{ a \cdot b \mid a \in A, b \in B \}$

$$A^{-1} = \{ a^{-1} : a \in A \}$$

$$A^n = A \cdot A^{n-1} \quad n \geq 2.$$

Proof:

(1) Recall that given  $f: X \rightarrow Y$  continuous,  $A \subset X$  then  $f(\overline{A}) \subset \overline{f(A)}$ .

Applying this to the multiplication map

$$m: G \times G \rightarrow G \\ (x, y) \mapsto x \cdot y$$

and the inversion

$$i: G \rightarrow G \\ x \mapsto x^{-1}$$

we get:

$$m(\overline{H} \times \overline{H}) = m(\overline{H \times H}) \subset \overline{m(H \times H)} \\ = \overline{H}.$$

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$$\text{And } i(\overline{H}) \subset \overline{i(H)} = \overline{H}$$

which shows (1).

(2) Let  $R \subset G$  be a complete set of representatives for  $G/H$ , that is:

$$G = \bigsqcup_{x \in R} x \cdot H = H \cup \bigsqcup_{x \in R \setminus \{e\}} L_x(H)$$

where  $\bigsqcup$  denotes disjoint union.

Since  $H$  is open and  $L_x : G \rightarrow G$  a

homeo., so is  $L_x(H)$  and hence

$\bigsqcup_{x \in R \setminus \{e\}} L_x(H)$  which implies that  $H$  is closed.

(3) Notice first that connected components are always closed. Thus the subset  $G_0$

is closed. Now  $m(G_0 \times G_0) \subset G$

and  $i(G_0) \subset G$  are connected subset

containing  $e \in G$ . Hence they are contained

in  $G_0$  which shows that  $G_0$  is a subgroup. Finally given  $g \in G$ , and observing that  $G \rightarrow G, x \mapsto gxg^{-1}$  is continuous we have that

$$\{gxg^{-1} : x \in G_0\}$$

is a connected subset of  $G$  containing  $e$  hence is contained in  $G_0$ . This

shows that  $G_0$  is normal in  $G$ .

(4) Observe that  $H := \bigcup_{n \geq 1} (V \cup V^{-1})^n$

is a subgroup of  $G$ . Now let  $e \in U \subset V$

be an open subset of  $G$ ; then  $U \subset H$

and since  $H$  is a group we have

$$L_h(U) \subset H \quad \forall h \in H.$$

But  $L_h(U) \ni h$  is an open neighborhood of  $h$ ; hence  $H$  is a neighborhood of

each of its points hence  $H$  is open.

By (2)  $H$  is hence closed; since  $G$  is connected and  $H \neq \emptyset$  we deduce that  $H = G$ . □

Remark 6.5 According to (3) we can write  $G = \bigsqcup_x G_0$  where the union is over a complete set of representatives of  $G/G_0$ . Hence the set  $\pi_0(G)$  of connected components of  $G$  acquires via its identification with  $G/G_0$  a group structure. It is an instructive exercise to compute  $\pi_0(G)$  in each of the examples 6.3.

### 6.3. Haar measure and convolution product.

Let now  $G$  be locally compact Hausdorff. The (arguably) most important fact about this class of groups is the existence and uniqueness of Haar measure which we now describe in more detail.

Let as usual  $C_0(G)$  be the  $\mathbb{C}$ -vector space of continuous compactly supported functions on  $G$ . Given any map  $F: G \rightarrow X$  into a set  $X$ , we denote  $\lambda(g)F: G \rightarrow X$  the map  $(\lambda(g)F)(x) = F(j_g^{-1}x)$ .

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Clearly if  $f \in C_0(G)$  then  $\lambda(g)f \in C_0(G)$  and one verifies easily that this way one obtains a group homomorphism

$$\lambda: G \rightarrow GL(C_0(G)).$$

The fundamental theorem is then:

Theorem 6.6. ~~Let~~ Let  $G$  be a locally compact Hausdorff group. Then there

exists a non-zero, positive linear

functional  $\Lambda: C_0(G) \rightarrow \mathbb{C}$

that is invariant under left translation,

i.e. 
$$\Lambda(\lambda(g)f) = \Lambda(f), \quad \forall g \in G$$
  
$$\forall f \in C_0(G).$$

Moreover given two such functionals

$\Lambda_1, \Lambda_2$  there exists  $c > 0$  with

$$\Lambda_2 = c \cdot \Lambda_1.$$

Such a functional is called a  $\mathbb{C}$ -left Haar-funct.

Using Riesz' representation theorem one obtains an equivalent formulation:

Corollary 6.7 There is, up to strictly positive scalar multiple, a unique non-zero, positive regular Borel measure  $\mu$  on  $G$  such that for every measurable set  $E \subset G$  and every  $g \in G$  :  $\mu(gE) = \mu(E)$ .

For the proof of Haar's theorem we refer e.g. to A. Weil, "L'intégration dans les groupes topologiques et ses applications".

A measure as in Corollary 6.7 is called a left Haar measure. In case  $G$  is

abelian we have  $L_g = R_g \quad \forall g \in G$  and

We call a left Haar measure just Haar measure.

## Examples 6.8

(1) The Lebesgue measure  $\mathcal{L}$  on  $\mathbb{R}$ , that is, the unique positive regular Borel measure on  $\mathbb{R}$  such that  $\mathcal{L}([a, b]) = b - a$  for  $a \leq b$  is a Haar measure for  $(\mathbb{R}, +)$  and so is  $\underbrace{\mathcal{L} \times \dots \times \mathcal{L}}_{n \text{ times}}$  for  $(\mathbb{R}^n, +)$ .

(2) A Haar measure for the multiplicative group  $(\mathbb{R}^x, \cdot)$  is given by

$$d\mu(x) = \frac{d\mathcal{L}(x)}{|x|}.$$

(3) Let  $G$  be discrete, then

$\mu(E) = \text{card}(E)$ ,  $\forall E \subseteq G$  is a left Haar measure.

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The uniqueness statement is probably more important than the existence statement, since it gives us immediately additional structure.

Corollary 6.9 Let  $\text{Aut}(G)$  denote the group of continuous automorphisms of  $G$ . Then there is a well defined group homomorphism

$$\text{mod}_G : \text{Aut}(G) \rightarrow \mathbb{R}_{>0}$$

into the multiplicative group  $(\mathbb{R}_{>0}, \cdot)$

such that for any left Haar functional

$$\lambda : \quad \lambda(f \circ \alpha^{-1}) = \text{mod}_G(\alpha) \lambda(f)$$

$$\forall f \in C_0(G)$$

$$\forall \alpha \in \text{Aut } G.$$

Proof. Observe that  $f \mapsto f \circ \alpha^{-1}$  is a linear map on  $C_0(G)$  preserving positivity

In addition we have for  $\alpha \in \text{Aut } G$ ,

$f \in C_0(G)$  and  $g, x \in G$ :

$$\begin{aligned} (\lambda(g)f)(\alpha^{-1}(x)) &= f(g^{-1}\alpha^{-1}(x)) = f(\alpha^{-1}(\alpha(g)_{sc}^{-1})) \\ &= \lambda(\alpha(g)) (f \circ \alpha^{-1})(x) \end{aligned}$$

And as a result if we define

$$\Lambda_x^\alpha(f) := \Lambda(f \circ \alpha^{-1})$$

then  $\Lambda_x^\alpha$  is a nonzero positive functional

$$\text{and } \Lambda_x^\alpha(\lambda(g)f) = \Lambda((\lambda(g)f) \circ \alpha^{-1})$$

$$= \Lambda(\lambda(\alpha(g)) (f \circ \alpha^{-1})) = \Lambda(f \circ \alpha^{-1}) = \Lambda_x^\alpha(f)$$

and hence  $\Lambda_x^\alpha$  is a Haar functional.

By uniqueness there is a constant

$$c_\alpha(x) > 0 \quad \text{with} \quad \Lambda_x^\alpha = \frac{c_\alpha(x)}{\lambda} \Lambda.$$

One verifies easily that  $C_\lambda(x)$  is independent on the choice of  $\lambda$ , and defines

$$\text{mod}_G(x) = C_\lambda(x).$$

Then from  $\Lambda_{x_1, x_2} = (\Lambda_{x_1})_{x_2}$

$$= C(x_2) \Lambda_{x_1} = C(x_2) C(x_1) \Lambda$$

and the independence of  $C_\lambda$  on  $\lambda$

follows:  $\text{mod}_G(x_1, x_2) = \text{mod}_G(x_1) \text{mod}_G(x_2)$ . □

### Example 6.10.

(1) Let  $G = (\mathbb{R}^n, +)$  then (exercise)

$\text{Aut } G = \text{GL}(n, \mathbb{R})$  and:

$$\text{mod}_G(x) = \frac{1}{|\det x|}$$

Indeed if  $\mathcal{L}$  is Lebesgue measure on  $\mathbb{R}^n$  and  $\alpha \in \text{GL}(n, \mathbb{R})$  then

$$\mathcal{L}(\alpha([0,1]^n)) = |\det \alpha| \cdot \mathcal{L}([0,1]^n).$$

This translates into the statement

$$\int_{[0,1]^n} \chi(\alpha^{-1}(x)) d\mathcal{L}(x) = |\det \alpha| \int_{[0,1]^n} \chi(x) d\mathcal{L}(x)$$

but the left hand side equals

$$\text{mod}_G(\alpha^{-1}) \int_{[0,1]^n} \chi(x) d\mathcal{L}(x).$$

(2) Let  $K$  be a locally compact field.

One gets this way two locally compact

(abelian) groups namely  $(K, +)$  and

$(K^\times, \cdot)$  which both have Haar

measures. Now with the identification

$$K^\times = GL(1, K) \hookrightarrow \text{Aut}(K, +)$$

We obtain a canonical homomorphism

$$\text{mod}_K : K^\times \longrightarrow \mathbb{R}_{>0}$$

Which if  $\mu$  is any Haar measure on  $(k, +)$  verifies  $\forall E \subset k$  Borel subset:

$$\mu(y \cdot E) = \text{mod}_k(y) \mu(E) \quad \forall y \in k^\times.$$

A non-trivial fact is that if  $k$  is non-discrete then  $y \mapsto \text{mod}_k(y)$

is not identically  $= 1$  and behaves like an absolute value, that is  $\exists c > 0$

such that  $\text{mod}_k(y_1 + y_2) \leq c \cdot \max(\text{mod}_k(y_1), \text{mod}_k(y_2))$ .

This is the starting point of the classification of non-discrete locally compact fields,

which can be found in A. Weil,

"Basic Number Theory" Chap I.

(3) It is not so easy to write down a Haar measure for  $(\mathbb{Q}_p, +)$ . However it is not difficult to determine

$\text{mod}(y)$  for  $y \in \mathbb{Q}_p$ . Indeed

We may assume  $y \in \mathbb{Z}_p$ , since  $\text{mod}(y) = \text{mod}(y\bar{y}^{-1})$ . Write  $y = p^n \cdot u$ ,  $n \geq 0$

and  $u \in \mathbb{U}$  invertible in  $\mathbb{Z}_p$ . Now

$\mathbb{Z}_p$  is open in  $\mathbb{Q}_p$  and compact so

if  $\mu$  is a Haar measure on  $\mathbb{Q}_p$ ,

$\mu(\mathbb{Z}_p) > 0$ . On one hand we

have  $\mu(y \cdot \mathbb{Z}_p) = \text{mod}(y) \mu(\mathbb{Z}_p)$

on the other:  $y \mathbb{Z}_p = p^n \mathbb{Z}_p$  and

the latter is the kernel of the surjective

hom.  $\Sigma_n : \mathbb{Z}_p \rightarrow A_n = \mathbb{Z}/p^n \mathbb{Z}$ .

Let then  $R$  be a complete set of  
representatives of  $\mathbb{Z}_p / p^n \mathbb{Z}_p$  i.e.  
 $|R| = p^n$  and  $\mathbb{Z}_p = \bigsqcup_{r \in R} (r + p^n \mathbb{Z}_p)$

which implies

$$\begin{aligned} \mu(\mathbb{Z}_p) &= \sum \mu(r + p^n \mathbb{Z}_p) \\ &= p^n \cdot \mu(p^n \mathbb{Z}_p) \end{aligned}$$

Hence  $\mu(p^n \mathbb{Z}_p) = p^{-n} \mu(\mathbb{Z}_p)$

and hence  $\text{mod}(y) = p^{-n}$ .