

and the spectrum

$$\text{Sp}([f]) = \text{Ess Ran}(f). \\ L^\infty(E)$$

It is customary to write f for an element $[f]$ of $L^\infty(E)$.

The next result shows how to use E to identify $L^\infty(E)$ with an abelian sub- C^* -algebra of $\mathcal{L}(\mathcal{H})$.

Thm 5.16 Given a resolution of the identity $E: \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ there is a C^* -algebra isomorphism

$$\psi: L^\infty(E) \rightarrow A$$

onto a C^* -subalgebra $A \subset \mathcal{L}(\mathcal{H})$

which is related to E by the formula:

$$\langle \mathcal{Y}(f) | x, y \rangle = \int_X f dE_{x, y}$$

where $x, y \in \mathcal{H}$, $f \in L^\infty(E)$.

$$\text{Moreover: } \|\mathcal{Y}(f)x\|^2 = \int_X |f|^2 dE_{x, x}$$

and an operator $Q \in \mathcal{L}(\mathcal{H})$ commutes with every $E(\omega) \iff Q$ commutes with A .

We will use the notation.

$$\mathcal{Y}(f) = \int_X f dE$$

Definition 5.17 A simple function on X is a function $S \in B^\infty(X)$ taking finitely many values.

Let $\mathcal{S}(X)$ be the \mathbb{C} -vector space

of simple functions on X . Clearly $\mathcal{J}(X)$ is a subalgebra of $\mathcal{B}^\infty(X)$ that is dense for the $\|\cdot\|$ -topology.

Proof of Thm 5.16 :

Let $s \in \mathcal{J}(X)$, then if $\alpha_1, \dots, \alpha_n$ are the distinct values of s we have

$$s = \sum_{i=1}^n \chi_{\omega_i} \alpha_i$$

where $\omega_i = s^{-1}(\alpha_i) \in \mathcal{B}$. We

define $\psi(s) := \sum_{i=1}^n \alpha_i E(\omega_i)$.

Then $\psi: \mathcal{J}(X) \rightarrow \mathcal{L}(\mathcal{H})$ is a

\ast -algebra map :

$$(1) \psi(s)^\ast = \sum \bar{\alpha}_i E(\omega_i)^\ast = \sum \bar{\alpha}_i E(\omega_i)$$

$= \gamma(\bar{s})$ since every $E(w)$ is self-adjoint.

$$(2) \text{ Let } s = \sum_{i=1}^n \alpha_i X_{w_i}$$

$$t = \sum_{j=1}^m \beta_j X_{w'_j}$$

Then:

$$\gamma(s)\gamma(t) = \sum_{i,j} \alpha_i \beta_j E(w_i) E(w'_j)$$

$$= \sum_{i,j} \alpha_i \beta_j E(w_i \cap w'_j)$$

Now observe:

$$s \cdot t = \sum_{i,j} \alpha_i \beta_j X_{w_i} \cdot X_{w'_j}$$

$$= \sum_{i,j} \alpha_i \beta_j X_{w_i \cap w'_j}$$

which implies $\gamma(s)\gamma(t) = \gamma(s \cdot t)$.

(3) Analogously one shows $\gamma(\alpha s + \beta t)$
 $= \alpha \gamma(s) + \beta \gamma(t)$.

(4) We have for $x, y \in \mathcal{H}$

$$\begin{aligned} \langle \gamma(s), x, y \rangle &= \sum_{i=1}^n \alpha_i \langle E(\omega_i) x, y \rangle \\ &= \sum_{i=1}^n \alpha_i E_{x, y}(\omega_i) \\ &= \int_X s \, dE_{x, y} \end{aligned}$$

Where $dE_{x, y}$ refers to the complex measure associated to $E_{x, y}$.

Next:

$$\gamma(s)^* \gamma(s) = \gamma(\bar{s}) \gamma(s) = \gamma(|s|^2)$$

and hence

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$$\begin{aligned}\|\psi(s)x\|^2 &= \langle \psi(s)^\dagger \psi(s)x, x \rangle \\ &= \langle \psi(|s|^2)x, x \rangle \\ &= \int_X |s|^2 dE_{x,x}.\end{aligned}$$

This implies

$$\begin{aligned}\|\psi(s)x\| &\leq \|s\|_\infty \cdot \sqrt{E_{x,x}(x)} \\ &= \|s\|_\infty \cdot \|x\|.\end{aligned}$$

Moreover if $x \in \text{Im}(E(\omega_j))$ then

$$\psi(s)x = \alpha_j E(\omega_j)x = \alpha_j \cdot x$$

Since the projectors $E(\omega_1), \dots, E(\omega_n)$ have mutually orthogonal range. Thus

if we choose j so that $|\alpha_j| = \|s\|_\infty$

we conclude $\|\psi(s)x\| = \|s\|_\infty \cdot \|x\|$

and hence $\|\mathcal{T}(s)\| = \|s\|_\infty$ (*).

Let now $f \in \mathcal{B}^\infty(X)$ and $(s_n)_{n \geq 1}$ a sequence in $\mathcal{P}(X)$ with

$$\lim_{n \rightarrow \infty} \|f - s_n\| = 0.$$

Then it follows from (*) that

$(\mathcal{T}(s_n))_{n \geq 1}$ forms a Cauchy sequence in $\mathcal{L}(X)$; let $\mathcal{T}(f) = \lim_{n \rightarrow \infty} \mathcal{T}(s_n)$.

One verifies easily that $\mathcal{T}(f)$ does not depend on the choice of $(s_n)_{n \geq 1}$.

Clearly (*) implies:

$$\|\mathcal{T}(f)\| = \|f\|_\infty.$$

Now let $x \in X$:

$$\langle \mathcal{T}(f)x, x \rangle = \lim_{n \rightarrow \infty} \langle \mathcal{T}(s_n)x, x \rangle$$

$$= \lim_X \int S_n dE_{x,x}$$

And since $\lim \|S_n - f\| = 0$ we get

$$= \int f dE_{x,x}$$

which implies

$$\langle \psi(f)_{x,y} \rangle = \int f dE_{x,y}.$$

An analogous argument shows

$$\|\psi(f)_x\|^2 = \int |f|^2 dE_{x,x}.$$

Thus ψ is a C^* -injection of $L^\infty(E)$

into $\mathcal{L}(\mathcal{H})$; its image A is therefore closed.

The final assertion is clear by approximation.

□