

Now we turn to a consequence of the spectral theorem which says that any normal operator is up to Hilbert space isomorphism given by a multiplication operator on an L^2 -space.

More generally :

Theorem 5.20

Let \mathcal{H} be a separable Hilbert space and $A \subset \mathcal{L}(\mathcal{H})$ an abelian sub- C^* -algebra containing $\text{Id}_{\mathcal{H}}$. Then there is a finite positive regular Borel measure μ on $\hat{A} \times N$ and a Hilbert space isomorphism

$$\lambda: \mathcal{H} \rightarrow L^2(\hat{A} \times N, \mu)$$

- 5 - 56 -

such that for all $T \in A$ and $\omega \in \kappa$:

$$\Lambda(T, \omega)(x, n) = \hat{T}(x) \Lambda(\omega)(x, n).$$

In other words if $M_{\hat{T}} : L^2(\hat{A} \times \mathbb{N}, \mu) \rightarrow$

denotes the multiplication operator

defined for $f \in L^2(\hat{A} \times \mathbb{N}, \mu)$

by $(M_{\hat{T}} f)(x, n) = \hat{T}(x) f(x, n)$

the following diagram commutes:

$$\begin{array}{ccc} x & \xrightarrow{\Lambda} & L^2(\hat{A} \times \mathbb{N}, \mu) \\ T \downarrow & & \downarrow M_{\hat{T}} \\ x & \xrightarrow{\Lambda} & L^2(\hat{A} \times \mathbb{N}, \mu) \end{array}$$

Let $A \subset L(\mathcal{H})$ be a (not-necessarily abelian) sub- C^* -algebra; then for every $v \in \mathcal{H}$, $A \cdot v \subset \mathcal{H}$ is an A -invariant vector subspace and so is its closure $\overline{A \cdot v}$.

Lemma 5.21 Let \mathcal{H} be a separable Hilbert space and $A \subset L(\mathcal{H})$ a sub- C^* -algebra. Then there exists a finite or countably infinite family of vectors v_1, v_2, \dots such that the closed subspaces $\overline{Av_i}$ are pairwise orthogonal and

$$\mathcal{H} = \hat{\bigoplus}_{i \geq 1} \overline{Av_i}$$

where $\hat{\oplus}$ denotes direct orthogonal sum.

Proof: (sketch) Consider the set

$$\mathcal{G} = \left\{ F \subset \mathcal{K} : \forall v + w \text{ in } F, \overline{A_v} \perp \overline{A_w} \right\}$$

then it is easy to see by Zorn's lemma that there is a maximal element $F \in \mathcal{G}$.

Then $\hat{\bigoplus}_{v \in F} \overline{A_v} = \mathcal{K}$: indeed

$\mathcal{L} := \hat{\bigoplus}_{v \in F} \overline{A_v}$ is A -invariant and

\mathcal{L}^\perp as well because A contains the adjoint of every element of A . By maximality, $\mathcal{L}^\perp = (0)$. Finally

observe that since \mathcal{K} is separable, F is ~~at most~~ countable.

[3]

Proof of Thm 5.20.

Let E be the resolution of identity on \hat{A} given by Thm 5.18. Let $v \in E$ $v \neq 0$. Then:

$$\langle T_v, v \rangle = \int_{\hat{A}} \hat{T} dE_{v,v}$$

$$\text{and hence } \|T_v\|^2 = \int_{\hat{A}} |\hat{T}|^2 dE_{v,v}.$$

So we have $T_v = 0$ iff $\hat{T} = 0$ almost everywhere wrt $E_{v,v}$.

We obtain

a well defined map

$$\begin{aligned} \mathcal{L}_v : A \cdot v &\longrightarrow L^2(\hat{A}, E_{v,v}) \\ T \cdot v &\longmapsto \hat{T} \end{aligned}$$

which is isometric and hence extends

- 5 - GO -

to a Hilbert space isomorphism

$$L : \overline{A^{\circ}} \rightarrow L^2(\hat{A}, E_{\circ, \nu})$$

Since $C(\hat{A})$ is dense in $L^2(\hat{A}, E_{\circ, \nu})$.

In addition $\forall a, T \in A$:

$$L_\nu(aT \cdot u) = \hat{a} \cdot \hat{T}.$$

and hence $\forall w \in \overline{A^{\circ}}$:

$$L_\nu(a \cdot w) = \hat{a} \cdot L_\nu(w).$$

Let now v_1, v_2, \dots be a sequence

of vectors, ($v_i \neq v_j$, $i \neq j$) such

that

$$\mathcal{K} = \bigoplus_{i \geq 1} \overline{A^{\circ} v_i}$$

is an orthogonal decomposition.

We can scale the v_i 's such that

- 5 - 61 -

$$\sum_{i \geq 1} \|v_i\|^2 < +\infty.$$

Define then the measure μ on
 $\hat{A} \times W$ by :

$$\mu = \sum_{n \geq 1} E_{v_n, u_n} \otimes \delta_n$$

Then μ is a positive regular Borel
measure on $\hat{A} \times W$ and

$$\begin{aligned} \mu(\hat{A} \times W) &= \sum_{n \geq 1} E_{v_n, u_n}(\hat{A}) \\ &= \sum_{n \geq 1} \|b_n\|^2 < +\infty. \end{aligned}$$

Then $A^{\#} = \bigoplus_{n \geq 1} L_{v_n} : \bigoplus_{n \geq 1} \overline{Av_n} \rightarrow L^2(\hat{A} \times W)$

is a Hilbert space isomorphism satisfying
the conclusions of Thm 5.20. \square