ETH Zürich, Spring 2022
D-MATH
Prof. Dr. Martin Schweizer Coordinator: David Martins

Brownian Motion and Stochastic Calculus

Exercise sheet 0

This exercise sheet is optional and the solutions are not to be submitted. These exercises may be helpful with recalling topics from the "Probability Theory" course, or in (quickly) learning them in preparation for the present course.

Exercise 0.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $Y_1, Y_2, \dots$ a sequence of independent random variables such that $E[Y_k] = 0$ and $E\big[|Y_k|^\alpha\big] \le \frac{1}{k^\beta}$ for some $\alpha > 1$ and $\beta > 0$.

(a) Show that the process $(S_k)_{k\ge 0}$ defined by $S_k := \sum_{j=1}^k Y_j$ is a martingale with respect to its natural filtration.

(b) Give a sufficient condition on α and β such that (S_k) converges P-almost surely as $k \to \infty$. For which $p \ge 1$ does it follow that (S_k) converges in L^p as $k \to \infty$?

(c) For $M > 0$, let τ_M be the hitting time defined by

$$\tau_M := \inf\{k \ge 0 : |S_k| \ge M\}.$$

Assuming the condition from (b), show that $P[\tau_M < \infty] = O(M^{-\alpha})$ as $M \to \infty$.

Exercise 0.2 Let X be a real-valued random variable with standard normal distribution as law and Y a random variable independent of X with law defined by

$$P[Y = 1] = p \quad \text{and} \quad P[Y = -1] = 1 - p, \qquad (0 \le p \le 1).$$

We define $Z := XY$.

(a) What is the law of Z? Is the vector (X, Z) a Gaussian vector?

(b) Calculate $\mathrm{Cov}(X, Z)$. For which $p \in [0, 1]$ are the random variables X and Z uncorrelated, i.e. $\mathrm{Cov}(X, Z) = 0$?

(c) Show that for each $p \in [0, 1]$, the random variables X and Z are *not* independent.

Exercise 0.3 We consider several examples of weak convergence. Results related with characteristic functions may be helpful with the proofs.

(a) Construct a sequence of rescaled binomial random variables X_n and a standard normal random variable X such that $X_n \Rightarrow X$ as $n \to \infty$.

(b) Construct a sequence of rescaled binomial random variables X_n and a Poisson random variable X such that $X_n \Rightarrow X$ as $n \to \infty$.

(c) Construct a sequence of rescaled geometric random variables X_n and an exponential random variable X such that $X_n \Rightarrow X$ as $n \to \infty$.

(d) Let X be a real-valued random variable with distribution function F. Construct a sequence of random variables X_n such that $X_n \Rightarrow X$ as $n \to \infty$ and each X_n has a continuous density function f_n.