Brownian Motion and Stochastic Calculus

Exercise sheet 1

Exercise 1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables and suppose that Z is $\sigma(X, Y)$-measurable. Use the monotone class theorem to show that there exists a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $Z = f(X, Y)$.

Hint: It may be helpful to start by assuming that Z is bounded.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P[Z_t = Z'_t] = 1, \ \forall\, t \geq 0$, while Z and Z' are *indistinguishable* if $P[Z_t = Z'_t, \ \forall\, t \geq 0] = 1$.

(a) Assume that X and Y are both right-continuous or both left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

Remark: A stochastic process is said to *have the path property* $\mathcal{P}$ ($\mathcal{P}$ can be continuity, right-continuity, differentiability, ...) if the property $\mathcal{P}$ holds for P-almost every path.

(b) Give an example showing that one of the implications of part (a) does not hold for general X, Y.

Exercise 1.3 Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The aim of this exercise is to show the following chain of implications:

$$X \text{ optional} \Rightarrow X \text{ progressively measurable} \Rightarrow X \text{ product-measurable and adapted.}$$

(a) Show that every progressively measurable process is product-measurable and adapted.

(b) Assume that X is adapted and *every* path of X is right-continuous. Show that X is progressively measurable.
Remark: The same conclusion holds true if every path of X is left-continuous.
Hint: For fixed $t \geq 0$, consider an approximating sequence of processes Y^n on $\Omega \times [0, t]$ given by $Y^n_0 = X_0$ and $Y^n_u = \sum_{k=0}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$.

(c) Recall that the optional σ-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.

(d) Use the monotone class theorem to show that every optional process is progressively measurable.

Exercise 1.4

(a) Let $(\Omega, \mathcal{F}, P)$ be a probability space and B a Brownian motion on $[0, 1]$. Let $k \in \mathbb{N}$ and

$$0 = s_1 < t_1 < s_2 < t_2 < \ldots < t_k < s_{k+1} = 1.$$

Find the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditional on $(B_{s_1}, \ldots, B_{s_{k+1}})$.

(b) Let $\mathcal{D} := \{a2^{-m} : m \in \mathbb{N}, a \in \{0, 1, \ldots, 2^m\}\}$. Let $Z_1, Z_2, \ldots$ be an infinite sequence of i.i.d. standard normal random variables. Construct in terms of the Z_j a stochastic process $(W_t)_{t\in\mathcal{D}}$ such that the law of W is equal to the law of $(B_t)_{t\in\mathcal{D}}$.