Brownian Motion and Stochastic Calculus

Exercise sheet 10

Exercise 10.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space satisfying the usual conditions.

(a) Let $W, \tilde{W}$ be two Brownian motions with respect to P and $(\mathcal{F}_t)_{t\geq 0}$. Show that we have $d\langle W, \tilde{W}\rangle_t = \rho_t\, dt$ for some predictable process ρ taking values in $[-1, 1]$.

Hint: Use the Kunita–Watanabe decomposition.

(b) The filtration $\mathbb{F}$ is called *P-continuous* if all local $(P, \mathbb{F})$-martingales are continuous. Show that $\mathbb{F}$ is P-continuous if and only if $\mathbb{F}$ is Q-continuous for all $Q \approx P$.

(c) Suppose that $\mathbb{F}$ is P-continuous and let $S = (S_t)_{t\geq 0}$ be a local $(Q, \mathbb{F})$-martingale for some $Q \approx P$. Show that S is a continuous P-semimartingale of the form

$$S = S_0 + M + \int \alpha\, d\langle M\rangle \tag{1}$$

for some $M \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$ and some $\alpha \in L^2_{\mathrm{loc}}(M)$.

Hint: Use Girsanov's theorem to find a semimartingale decomposition for S under P. Then use the Kunita–Watanabe decomposition under P to describe its finite variation part.

Remark: If S has the form (1), one says that it satisfies the *structure condition SC*. This is a useful concept in mathematical finance.

Exercise 10.2 Let $B = (B^1, B^2, B^3)$ be a Brownian motion in $\mathbb{R}^3$ and $Z = (Z^1, Z^2, Z^3)$ a standard normal random variable. Define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{|Z + B_t|}.$$

Note that $P[B_t \neq x, \forall t \geq 0] = 1$ for any $x \in \mathbb{R}^3 \setminus \{0\}$; see Exercise **9.4**.

(a) Show that $P[B_t \neq -Z, \forall t \geq 0] = 1$, so that M is a.s. well defined.

(b) Show that $|Z + B_t|^2 \sim \mathrm{Gamma}(\frac{3}{2}, \frac{1}{2(t+1)})$ for each $t > 0$, i.e., its density is given by

$$f_t(y) = \frac{(2(t+1))^{-3/2} y^{1/2}}{\Gamma(3/2)} \exp\left(-\frac{y}{2(t+1)}\right), \quad y \geq 0.$$

(c) Show that M is a continuous local martingale. Moreover, show that M is bounded in L^2, i.e., $\sup_{t\geq 0} E[|M_t|^2] < \infty$.

(d) Show that M is a *strict local martingale*, i.e., M is not a martingale.

Hint: Show that $E[M_t] \to 0$ as $t \to \infty$.

Remark: This is the standard example of a local martingale which is not a (true) martingale. It also shows that even boundedness in L^2 (which implies uniform integrability) does not guarantee the martingale property.

Exercise 10.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the raw filtration generated by W. Moreover, fix $T > 0$, $a < b$, and let $F := \mathbb{1}_{\{a\leq W_T \leq b\}}$. The goal of this exercise is to find explicitly the integrand $H \in L^2_{\mathrm{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^\infty H_s \, dW_s. \qquad (\star)$$

(a) Show that the martingale $M = (M_t)_{t\geq 0}$ given by $M_t := E[F|\mathcal{F}_t]$ has the representation

$$M_t = g(W_t, t), \quad 0 \leq t < T,$$

for a C^2 function $g : \mathbb{R} \times [0, T) \to \mathbb{R}$. Compute g in terms of the distribution function Φ of the standard normal distribution.

(b) Let (t_n) be a sequence of times such that $t_n \nearrow T$. Use Itô's formula to find predictable processes H^n such that

$$M^{t_n} - M_0 = H^n \bullet W, \quad \text{for each } n \in \mathbb{N}.$$

Hint: Since M is a martingale, you do not need to calculate all the terms in Itô's formula.

(c) Find H such that $(\star)$ holds.