Brownian Motion and Stochastic Calculus

Exercise sheet 11

Exercise 11.1 Let $\theta \in \mathbb{R}, \sigma > 0$ and $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

(a) Find a strong solution to the *Langevin equation*

$$dX_t = -\theta X_t\, dt + \sigma\, dW_t, \quad X_0 = x \in \mathbb{R}.$$

Hint: Consider $U_t = e^{\theta t} X_t$.

Remark: For $\theta > 0$, this SDE describes exponential convergence to zero "with noise".

(b) Show that there exists a Brownian motion B such that $Y := X^2$ satisfies the SDE

$$dY_t = (-2\theta Y_t + \sigma^2)\, dt + 2\sigma\sqrt{Y_t}\, dB_t. \tag{$\star$}$$

In other words, show that $(\Omega, \mathcal{F}, \mathbb{F}, P, B, Y)$ is a weak solution of the SDE $(\star)$.

Exercise 11.2 Let $(W_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$dX_t = \left(\sqrt{1 + X_t^2} + \frac{1}{2}X_t\right) dt + \sqrt{1 + X_t^2}\, dW_t, \quad X_0 = x \in \mathbb{R}. \tag{1}$$

(a) Show that for any $x \in \mathbb{R}$, the SDE defined in (1) has a unique strong solution.

(b) Show that $(X_t)_{t\geq 0}$ defined by $X_t = \sinh\left(\sinh^{-1}(x) + t + W_t\right)$ is the unique solution of (1).

Hint: Consider the process $(Y_t)_{t\geq 0}$ defined by $Y_t := \sinh^{-1}(X_t)$.

Exercise 11.3

(a) Let $x_0 \in \mathbb{R}$, W be a Brownian motion and $(\mu_t)_{t\geq 0}$ a bounded predictable process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Show that there exists a unique adapted solution $(X_t)_{t\geq 0}$ to the equation

$$X_t = x_0 + \int_0^t \mu_s ds + \int_0^t X_s dW_s,$$

which is given by

$$X_t = \mathcal{E}(W)_t \left(x_0 + \int_0^t \big(\mathcal{E}(W)_s\big)^{-1} \mu_s ds \right).$$

In particular, if $x_0 \geq 0$ and $\mu_t \geq 0$ for all $t \geq 0$, then $X_t \geq 0$ for all $t \geq 0$.

(b) Let $x_1, x_2 \in \mathbb{R}$ and $a_1, a_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous functions that are Lipschitz and have linear growth, as in Theorem (4.7.4) of the lecture notes. Suppose that $x_1 \geq x_2$ and $a_1(t,x) \geq a_2(t,x)$ for all $t \geq 0, x \in \mathbb{R}$. Show that there exist unique solutions X^1 and X^2 to the SDEs

$$X_t^1 = x_1 + \int_0^t a_1(s, X_s^1) ds + \int_0^t X_s^1 dW_s,$$
$$X_t^2 = x_2 + \int_0^t a_2(s, X_s^2) ds + \int_0^t X_s^2 dW_s,$$

and that $X_t^1 \geq X_t^2$ for all $t \geq 0$ almost surely,

Hint: Argue that $a_2(s, X_s^1) - a_2(s, X_s^2) = \pi_s(X_s^1 - X_s^2)$, where π is a predictable process bounded by K (the Lipschitz constant). A Girsanov transformation may also be useful.

Exercise 11.4

(a) Let W be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and $\bar{f} : \mathbb{R} \to \mathbb{R}$ be a measurable bounded odd function. That is, $\bar{f}(-x) = -\bar{f}(x)$ or, equivalently,

$$\bar{f}(x) = f(x)\mathbb{1}_{\{x>0\}} - f(-x)\mathbb{1}_{\{x<0\}},$$

for some bounded measurable function $f : (0, \infty) \to \mathbb{R}$. Show that the process

$$Y = \bar{f}(W) \bullet W$$

is adapted with respect to the P-augmented filtration $\mathbb{F}^{|W|}$ generated by $|W|$ and all nullsets.

Hint: Start by considering $f(x) = \sin(\lambda x)$ for $\lambda > 0$ and applying Itô's theorem. Conclude by approximation.

(b) Let X be a Brownian motion and define $B = \text{sign}(X) \bullet X$, so that B is a Brownian motion and the process $X = \text{sign}(X) \bullet B$ solves the Tanaka equation as in Example (4.7.10). Show that X is not adapted to the filtration generated by B.