Brownian Motion and Stochastic Calculus

Exercise sheet 12

Exercise 12.1 Consider the SDE

$$\begin{aligned} dX_t^x &= a(X_t^x)\,dt + b(X_t^x)\,dW_t, \\ X_0^x &= x, \end{aligned} \tag{1}$$

where W is an $\mathbb{R}^m$-valued Brownian motion and the functions $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable and locally bounded (i.e., bounded on compact sets). Let $U \subseteq \mathbb{R}^d$ be a bounded open set such that the stopping time $T_U^x := \inf\{s \geq 0 : X_s^x \notin U\}$ is P-integrable for all $x \in U$. Consider the boundary problem

$$\begin{aligned} Lu(x) + c(x)u(x) &= -f(x) \qquad \text{for } x \in U, \\ u(x) &= g(x) \qquad \text{for } x \in \partial U, \end{aligned} \tag{2}$$

where $c, f \in C_b(U)$ and $g \in C_b(\partial U)$ are given functions such that $c \leq 0$ on U, and the linear operator L is defined by

$$Lf(x) := \sum_{i=1}^d a_i(x)\,\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^d \left(b\,b^\top\right)_{ij}(x)\,\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x).$$

Suppose that $(X_t^x)_{t\geq 0}$ solves the SDE (1) for some $x \in U$ and $u \in C^2(U) \cap C(\bar{U})$ is a solution to the boundary problem (2). Show that

$$u(x) = E\left[g(X_{T_U^x}^x)\,\exp\left(\int_0^{T_U^x} c(X_s^x)\,ds\right)\right] + E\left[\int_0^{T_U^x} f(X_s^x)\,\exp\left(\int_0^s c(X_r^x)\,dr\right)ds\right].$$

Exercise 12.2

(a) Show that the spaces $\mathcal{R}^2$ and $\mathcal{A}^2$ defined in page 174 are Banach spaces.

(b) Let β, γ be bounded predictable processes and define

$$Y = \mathcal{E}\left(-\int \beta_s ds - \int \gamma_s dW_s\right).$$

Show that, for any $T > 0$, the random variable

$$Y_T^* := \sup_{0\leq s\leq T} |Y_s|$$

is in L^p for every $p < \infty$ (see page 180).

Exercise 12.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the (raw) filtration generated by W. Let $T > 0$, $\alpha > 0$ and let F be a bounded, $\mathcal{F}_T$-measurable random variable.

(a) Show that the process $X = (X_t)_{0\leq t\leq T}$ given by

$$X_t = -\alpha \log E[\exp(-F/\alpha) \mid \mathcal{F}_t]$$

solves the BSDE

$$dX_t = \frac{1}{2\alpha} Z_t^2 dt + Z_t dW_t,$$
$$X_T = F.$$

Hint: We have that $X_t = -\alpha \log Y_t$, where $Y_t := E[\exp(-F/\alpha)|\mathcal{F}_t]$. Apply Itô's representation theorem to Y_T and Itô's formula to X to derive a solution pair $(X, Z) \in \mathcal{R}^2 \times L^2(W)$ for the BSDE.

Remark: Note that the generator of this BSDE is not Lipschitz, but quadratic in Z.

(b) Let $b \in \mathbb{R}$. Show that the process $X = (X_t)_{0 \le t \le T}$ given by

$$X_t = -\alpha \left(\frac{1}{2} b^2 (t - T) - b\, W_t + \log E[\exp(bW_T - F/\alpha) \mid \mathcal{F}_t] \right)$$

solves the BSDE

$$dX_t = \left(\frac{1}{2\alpha} Z_t^2 - b\, Z_t \right) dt + Z_t dW_t,$$
$$X_T = F.$$