Brownian Motion and Stochastic Calculus

Exercise sheet 13

Exercise 13.1 Let X be a Lévy process with values in $\mathbb{R}^d$ and $f_t(u) := E[e^{\mathrm{i}u^\top X_t}]$. Recall that X is stochastically continuous, i.e., the map $t \mapsto X_t$ is continuous in probability, and that $f_{t+s}(u) = f_t(u) f_s(u)$ and $f_0(u) = 1$ for all $s, t \geq 0$ and $u \in \mathbb{R}^d$.

(a) Show that $f_s(u)^n = f_{ns}(u)$ and $f_t(u) = f_{t/n}(u)^n$ for all $n \in \mathbb{N}$ and $s, t \geq 0$.

(b) Show that $t \mapsto f_t(u)$ is right-continuous and $f_t(u) \neq 0$ for all $t \geq 0$ and $u \in \mathbb{R}^d$.

(c) Fix $u \in \mathbb{R}^d$ and let $\tilde{z} \in \mathbb{C}$ be such that $f_1(u) = \exp(\tilde{z})$. Show that there exists a unique $\hat{k} \in \mathbb{Z}$ such that

$$f_{2^{-n}}(u) = \exp\left(\frac{\tilde{z} + 2\hat{k}\pi\mathrm{i}}{2^n}\right)$$

for each $n \in \mathbb{N}$.

(d) In the setup of (e), let $z := \tilde{z} + 2\hat{k}\pi\mathrm{i}$ and define the function $g(t) = \exp(tz)$ (which can be seen as a definition of $t \mapsto f_1(u)^t$). Show that $f_t(u) = g(t)$ for all $t \geq 0$.

Exercise 13.2

(a) Let N be a one-dimensional Poisson process and $(Y_i)_{i\geq 1}$ a sequence of i.i.d. $\mathbb{R}^d$-valued random variables independent of N. We define the *compound Poisson process* by $X_t := \sum_{j=1}^{N_t} Y_j$. Show that X is a Lévy process and calculate its Lévy triplet.

(b) Does there exist a Lévy process X such that X_1 is uniformly distributed on $[0, 1]$?

(c) Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be $\mathbb{R}^d$-valued processes such that the joint process (X, Y) is Lévy with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)$. Show that if $E[e^{\mathrm{i}u^\top X_t} e^{\mathrm{i}v^\top Y_t}] = E[e^{\mathrm{i}u^\top X_t}]\, E[e^{\mathrm{i}v^\top Y_t}]$ for all $u, v \in \mathbb{R}^d$ and $t \geq 0$, then X and Y are independent.

Exercise 13.3

(a) Let $\tilde{\nu}$ be a finite measure supported on $[\varepsilon, \infty)$ for some $\varepsilon > 0$, and $\tilde{\lambda} := \tilde{\nu}([\varepsilon, \infty)) > 0$. Suppose that (N_t) is a Poisson process with rate $\tilde{\lambda}$ and $(\tilde{Y}_n)$ are i.i.d. random variables with distribution $\tilde{\lambda}^{-1}\tilde{\nu}$. Check using Exercise **13.2(a)** that the process $J_t^{\tilde{\nu}} := \sum_{j=1}^{N_t} \tilde{Y}_j$ is a Lévy process with Lévy triplet $(\tilde{b}, 0, \tilde{\nu})$, where $\tilde{b} = \int \mathbb{1}_{\{|x|\leq 1\}} x\, d\tilde{\nu}$.

(b) Suppose that $\tilde{\nu}$ has compact support, i.e., $\tilde{\nu}((K, \infty)) = 0$ for some $K \in (0, \infty)$, so that $\tilde{\nu}([\varepsilon, K]) = \tilde{\lambda}$. Find a constant $\mu > 0$ such that the process $M^{\tilde{\nu}}$ defined by

$$M_t^{\tilde{\nu}} := J_t^{\tilde{\nu}} - \mu t$$

is a martingale. If $\tilde{\nu}$ is not compactly supported, under what assumption can we find such a constant μ?

(c) For some $K > 0$, let ν be a measure supported on $[0, K]$ such that $\nu(\{0\}) = 0$ and $\nu((\varepsilon, K]) < \infty$ for each $\varepsilon > 0$. Choose a sequence $(a_m)_{m\in\mathbb{N}_0}$ such that $a_0 = K$ and $a_m \searrow 0$, and let $(\nu_m)_{m\in\mathbb{N}}$ be a sequence of measures that are absolutely continuous with respect to ν with respective densities $\frac{d\nu_m}{d\nu} = \mathbb{1}_{(a_m, a_{m-1}]}$. As in (a), for each $m \in \mathbb{N}$, let $(N_t^m)_{t \geq 0}$ be a Poisson process with rate $C_m := \nu((a_m, a_{m-1}])$ and (Y_n^m) be i.i.d. random variables with distribution $C_m^{-1}\nu_m$. We suppose that the (N^m) and (Y_n^m) are all independent, and define J^{ν_m} and M^{ν_m} as in (a) and (b).

Show that for each $k \geq 1$, the process $J^k := \sum_{m=1}^k J^{\nu_m}$ is Lévy and find its Lévy triplet. Find a constant μ_k such that $M_t^k := J_t^k - \mu_k t$ is a martingale.

(d) Suppose that $\int_0^K x^2 \nu(dx) < \infty$. For any $T > 0$, show that the sequence of stopped martingales $((M^k)^T)_{k \in \mathbb{N}}$ converges in $\mathcal{H}_0^2$.

(e) Under the assumption in (d), does $(J^k)_{k \in \mathbb{N}}$ converge?