Brownian Motion and Stochastic Calculus

Exercise sheet 2

Exercise 2.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space, W a Brownian motion on $[0, \infty)$, Z a random variable independent of W and $t^* \in (0, \infty)$. We define another stochastic process $W' = (W'_t)_{t \geq 0}$ by

$$W'_t = W_t 1_{\{t < t^*\}} + \Big(W_{t^*} + Z(W_t - W_{t^*})\Big) 1_{\{t \geq t^*\}}.$$

Find all possible distributions of Z such that W' is a Brownian motion.

Exercise 2.2 Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with $X_0 = 0$ P-a.s., and let $\mathbb{F}^X = (\mathcal{F}^X_t)_{t \geq 0}$ denote the (raw) filtration generated by X, i.e., $\mathcal{F}^X_t = \sigma(X_s;\, s \leq t)$. Show that the following two properties are equivalent:

(i) X has *independent increments*, i.e., for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the increments $X_{t_i} - X_{t_{i-1}}$, $i = 1, \ldots, n$, are independent.

(ii) X has $\mathbb{F}^X$*-independent increments*, i.e., $X_t - X_s$ is independent of $\mathcal{F}^X_s$ whenever $t \geq s$.

Remark: This also shows the equivalence between the two definitions of Brownian motion with properties (BM2) and (BM2′), respectively, when we choose $\mathbb{F} = \mathbb{F}^W$.

Hint: For proving "(i) ⇒ (ii)", you can use the monotone class theorem. When choosing $\mathcal{H}$, recall that a random variable Y is independent of a σ-algebra $\mathcal{G}$ if and only if one has the product formula $E[g(Y)Z] = E[g(Y)]E[Z]$ for all bounded Borel-measurable functions $g : \mathbb{R} \to \mathbb{R}$ and all bounded $\mathcal{G}$-measurable random variables Z.

Exercise 2.3 A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder-continuous of order α at $x \in D$ if there exist $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $y \in D$ with $|x - y| \leq \delta$. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder-continuous of order α if it is locally Hölder-continuous of order α at each $x \in D$.

(a) Let $Z \sim N(0, 1)$. Prove that $P[|Z| \leq \varepsilon] \leq \varepsilon$ for any $\varepsilon \geq 0$.

(b) Prove that for any $\alpha > \frac{1}{2}$, P-almost all Brownian paths are nowhere on $[0, 1]$ locally Hölder-continuous of order α.
Hint: Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\big\{W_\cdot(\omega) \text{ is locally } \alpha\text{-Hölder at some } s \in [0, 1]\big\}$ is contained in the set $\bigcup_{C \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{n \geq m} \bigcup_{k=0,\ldots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C \frac{1}{n^\alpha} \right\}.$

(c) The *Kolmogorov–Čentsov theorem* states that an $\mathbb{R}$-valued process X on $[0, T]$ satisfying

$$E\big[|X_t - X_s|^\gamma\big] \leq C\,|t - s|^{1+\beta}, \quad s, t \in [0, T],$$

where $\gamma, \beta, C > 0$, has a version which is locally Hölder-continuous of order α for all $\alpha < \beta/\gamma$. Use this to deduce that Brownian motion is for every $\alpha < 1/2$ locally Hölder-continuous of order α.

Remark: One can also show that the Brownian paths are *not* locally Hölder-continuous of order 1/2. The exact modulus of continuity was found by P. Lévy.

Exercise 2.4

(a) Let W be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$ be the natural filtration of W. Let $\mathcal{F}_{0+} := \cap_{t>0}\mathcal{F}_t$. Show *Blumenthal's 0-1 law*: for $A \in \mathcal{F}_{0+}$, either $P[A] = 0$ or $P[A] = 1$.

Hint: Show that A and the increments of W are independent.

(b) Show that

$$P\left[\limsup_{t \searrow 0} \frac{W_t}{\sqrt{t}} = \infty\right] = 1.$$

Hint: Start by showing that for each $C > 0$,

$$\lim_{t \searrow 0} P\left[\sup_{0 \leq s \leq t} \left(W_s - C\sqrt{s}\right) > 0\right] > 0$$

and use (a).