Brownian Motion and Stochastic Calculus

Exercise sheet 3

Exercise 3.1 Given a measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, we set $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$ and define for any $\mathbb{F}$-stopping time τ the σ-field

$$\mathcal{F}_\tau := \big\{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \text{ for all } t \geq 0\big\}.$$

Let S, T be two $\mathbb{F}$-stopping times. Show that:

(a) if $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$, and in general, $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

(b) $\{S < T\}$, $\{S \leq T\}$ belong to $\mathcal{F}_S \cap \mathcal{F}_T$. Moreover, for any $A \in \mathcal{F}_S$, $A \cap \{S < T\}$ and $A \cap \{S \leq T\}$ belong to $\mathcal{F}_{S\wedge T}$.

(c) For any stopping time τ,

$$\mathcal{F}_\tau = \sigma(X_\tau : X \text{ an optional process}).$$

Exercise 3.2 Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the process X defined by

$$X_t := e^{-t} B_{e^{2t}}\,, \quad t \in \mathbb{R}.$$

(a) Show that $X_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{R}$.

(b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is *time reversible*, i.e. $(X_t)_{t\geq 0} \overset{(\mathrm{d})}{=} (X_{-t})_{t\geq 0}$.

Hint: Use the time inversion property of Brownian motion, i.e., if W is a Brownian motion,

$$\tilde{W}_t := \begin{cases} 0, & \text{if } t = 0, \\ tW_{1/t}, & \text{if } t > 0, \end{cases}$$

is also a Brownian motion.

Remark: The process X is called an Ornstein–Uhlenbeck process.

Exercise 3.3 Let W be a Brownian motion with respect to its natural filtration. Show that

$$M_t^{(1)} = e^{t/2} \cos W_t, \qquad M_t^{(2)} = tW_t - \int_0^t W_u du, \qquad M_t^{(3)} = W_t^3 - 3tW_t$$

are martingales.

Hint: You may want to use the formula for the characteristic function of a Gaussian random variable. A trigonometric identity for $\cos(a + b)$ may also be useful; alternatively, you may use that for independent random variables X and Y and if the density f_X exists, we have

$$E\big[g(X,Y) \mid Y\big] = \int_{\mathbb{R}} g(x,Y) f_X(x) dx$$

for any bounded measurable function $g : \mathbb{R}^2 \to \mathbb{R}$.

Exercise 3.4 Let $\rho \in (0,1)$. For a bounded measurable function $f : [0,1] \to \mathbb{R}$, set $f(t) = f(0)$ for $t < 0$ and define the moving average function $\mathrm{MA}_\rho f$ by

$$(\mathrm{MA}_\rho f)(t) = \frac{1}{\rho} \int_{t-\rho}^{t} f(u) du.$$

Define $\tau(f) = \inf\{t \geq 0 : f(t) \geq (\mathrm{MA}_\rho f)(t) + 1\} \wedge 1$. Show that if X^n is an approximation to a Brownian motion W as in Donsker's theorem, then $\tau(X^n) \to \tau(W)$ in distribution.