Brownian Motion and Stochastic Calculus

Exercise sheet 4

Exercise 4.1 Let W be a Brownian motion on $[0,\infty)$ and $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ constants. The stochastic process $S = (S_t)_{t\geq 0}$ given by

$$S_t = S_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big)$$

is called *geometric Brownian motion.*

(a) Prove that for $\mu \neq \sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = \infty \quad P\text{-a.s.} \qquad \text{or} \qquad \lim_{t\to\infty} S_t = 0 \quad P\text{-a.s.}$$

When do the respective cases arise?

(b) Discuss the behaviour of (S_t) as $t \to \infty$ in the case $\mu = \sigma^2/2$.

(c) Henceforth, suppose that $\mu = 0$. Show that S is a martingale, but not uniformly integrable.

(d) Let τ be a finite stopping time independent of W. Show that $E[S_\tau] = S_0$.

(e) Fix $S_0 = 1$, let $a \in (0,1)$ and let $\tau_a = \inf\{t : S_t \leq a\}$ be its hitting time. Show that $\tau_a < \infty$ almost surely and that $S_{\tau_a} = a < 1$. In particular, $E[S_{\tau_a}] = a < 1 = S_0$.

Exercise 4.2 Consider two stopping times σ, τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The goal of this exercise, together with exercise **3.1**, is to show that

$$E[E[\,\cdot\,|\mathcal{F}_\sigma]|\mathcal{F}_\tau] = E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}] = E[E[\,\cdot\,|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \quad P\text{-a.s.,} \qquad (\star)$$

i.e., the operators $E[\,\cdot\,|\mathcal{F}_\tau]$ and $E[\,\cdot\,|\mathcal{F}_\sigma]$ commute and their composition equals $E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.
Remark: For arbitrary sub-σ-algebras $\mathcal{G}, \mathcal{G}' \subseteq \mathcal{F}$, the conditional expectations $E[E[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $E[E[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $E[\,\cdot\,|\mathcal{G}\cap\mathcal{G}']$ do **not** coincide in general.

(a) Show that if Y is $\mathcal{F}_\sigma$-measurable, then $Y\mathbb{1}_{\{\sigma\leq\tau\}}$ and $Y\mathbb{1}_{\{\sigma<\tau\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(b) Show that $E[Y|\mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable if Y is an integrable $\mathcal{F}_\sigma$-measurable random variable. Conclude $(\star)$.

(c) Let $M = (M_t)_{t\geq 0}$ be a martingale with all trajectories right-continuous. Show that the stopped process $M^\tau = (M_{\tau\wedge t})_{t\geq 0}$ is again a martingale.
Hint: Use $(\star)$ and the stopping theorem.

Exercise 4.3 Let $(S, \mathcal{S})$ be a measurable space, let $Y = (Y_t)_{t\geq 0}$ be the canonical process on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$, i.e., $Y_t(y) = y(t)$ for $y \in S^{[0,\infty)}$, $t \geq 0$, and let $(K_t)_{t\geq 0}$ be a transition semigroup on $(S, \mathcal{S})$. Moreover, for each $x \in S$, assume that there exists a unique probability measure $\mathbb{P}_x$ on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ under which Y is a Markov process with transition semigroup $(K_t)_{t\geq 0}$ and initial distribution $\nu = \delta_{\{x\}}$.

Suppose $Z \geq 0$ is an $\mathcal{S}^{[0,\infty)}$-measurable random variable on $S^{[0,\infty)}$. Use the monotone class theorem to prove that the map $x \mapsto \mathbb{E}_x[Z], x \in S$, is $\mathcal{S}$-measurable.

Exercise 4.4 Part (a) of this exercise is optional, but the results are needed in (b) and (c).

(a) Let $L \in \mathbb{N}$ and consider a matrix $Q \in \mathbb{R}^{L\times L}$. For $t \in [0, +\infty)$, define $\exp(tQ) := \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}$.

1. Show that $\exp(tQ)$ is well-defined for any $t \in [0, +\infty)$ and $\exp(0Q) = \mathbb{I}$, the identity matrix.
2. Show that $Q^n \exp(tQ) = \exp(tQ)Q^n$ and that $\exp(sQ)\exp(tQ) = \exp(tQ)\exp(sQ) = \exp((t+s)Q)$ for any $n \in \mathbb{N}$ and $s, t \geq 0$.
3. Show that
$$\lim_{h\searrow 0} \frac{\exp((t+h)Q) - \exp(tQ)}{h} = Q\exp(tQ).$$
4. Show that
$$\lim_{M\to\infty} \left(1 + \frac{tQ}{M}\right)^M = \exp(tQ).$$

(b) Consider $S = \{x_1, \ldots, x_L\} \subseteq \mathbb{R}$. Define the operators $(K_t)_{t\geq 0}$ by
$$K_t(x_i, A) = \sum_{x_j\in A} (\exp(tQ))_{ij}, \qquad \text{for } A \subseteq S.$$

Show that $K_{s+t}(x_i, \{x_\ell\}) = \sum_{j=1}^{L} K_s(x_i, \{x_j\})K_t(x_j, \{x_\ell\})$ for $s, t \geq 0$.

(c) Suppose that there exists a Markov process X taking values in S such that
$$\mathbb{P}_{x_i}[X_t = x_j] = K_t(x_i, \{x_j\}) = (\exp(tQ))_{ij}. \tag{1}$$

Noting the fact that for all $t \geq 0$ and $x_i \in S$, the map $A \mapsto K_t(x_i, A)$ must then be a probability measure, what does this imply about Q?