Brownian Motion and Stochastic Calculus

Exercise sheet 5

Exercise 5.1

(a) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space. Assume that $\Omega = \{\omega_1, \ldots, \omega_k\}$ is finite and that $\mathcal{F} = 2^\Omega$.

Show that the $\mathbb{R}^k$-valued process

$$X_t = \big(P[\{\omega_1\} \mid \mathcal{F}_t], \ldots, P[\{\omega_k\} \mid \mathcal{F}_t]\big)^\top$$

is a Markov process.

(b) Let W be a Brownian motion. Which of the following processes X are Markov? Write down the corresponding transition kernels in those cases.

1. $X_t = |W_t|$ (reflected Brownian motion).
2. $X_t = \int_0^t W_u du$ (integrated Brownian motion).
3. $X_t = W_{\tau_a \wedge t}$, where $\tau_a = \inf\{t \geq 0 : W_t \geq a\}$ is the hitting time of $a > 0$.
4. $X_t = W_t^\tau$ for a random time $\tau \sim \operatorname{Exp}(1)$ independent of W.
5. $X_t = t - t \wedge \tau$, where $\tau \sim \operatorname{Exp}(1)$ is a random time.

Exercise 5.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion and define the *integrated Brownian motion* $Y = (Y_t)_{t\geq 0}$ by $Y_t = \int_0^t W_s ds$. Moreover, let $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\geq 0}$ be the raw filtration generated by W.

(a) For each $h \geq 0$, show that the pair (W_h, Y_h) has a two-dimensional normal distribution with mean zero and covariance matrix given by

$$\begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}.$$

Hint: You may want to apply Donsker's theorem by constructing a continuous mapping $F : C([0,\infty)) \to \mathbb{R}^2$ such that $F((W_t)_{t\geq 0}) = (W_h, Y_h)$. You may also use a result on weak convergence of Gaussian random variables.

(b) Show that the pair (W, Y) is a (homogeneous) Markov process with state space $\mathbb{R}^2$, filtration $\mathbb{F}^W = \mathbb{F}^{W,Y}$ and transition semigroup $(K_h)_{h\geq 0}$ given by

$$K_h\big((w,y),\cdot\big) = \mathcal{N}\left(\begin{pmatrix} w \\ y + hw \end{pmatrix}, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix} \right), \quad h \geq 0.$$

Exercise 5.3

(a) Recall the canonical space $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ of all real-valued functions equipped with the σ-algebra generated by all projections. Let $\lambda > 0$, $x \in \mathbb{R}$ and construct on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ a probability measure P such that the canonical process $(Y_t)_{t\geq 0}$ has independent increments, satisfies $P[Y_0 = x] = 1$ and $Y_t - Y_s \sim \operatorname{Poi}(\lambda(t-s))$.

Hint: Use the Kolmogorov consistency theorem.

(b) A Poisson process is a process $(N_t)_{t\geq 0}$ such that all trajectories are RCLL and piecewise constant, all jumps are of size $+1$, and the increments $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$ are independent. Show that the process $(Y_t)_{t\geq 0}$ defined in (a) admits a version which is a Poisson process.

(c) Let $(N_t)_{t\geq 0}$ be a Poisson process. For $n \in \mathbb{N}$, find the distribution of the random variables

$$\tau_n = \inf\{t \geq 0 : N_t - N_0 = n\}.$$

(d) Show that N is a Markov process.