Brownian Motion and Stochastic Calculus

Exercise sheet 6

Exercise 6.1 Let $(S, \mathcal{S}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and for each $x \in \mathbb{R}^2$, let $\mathbb{P}_x$ denote the unique probability measure on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ under which the coordinate process Y is a 2-dimensional Brownian motion starting at x. Show that for any $x \in \mathbb{R}^2$,

$$\mathbb{P}_x\left[\sup\{t \geq 0 : Y_t \in \mathcal{O}\} = \infty \text{ for every non-empty open set } \mathcal{O} \subseteq \mathbb{R}^2\right] = 1.$$

Hint: For any $x \in \mathbb{R}^2$ and $r \geq 0$, we define $\overline{B}(x,r) := \{y \in \mathbb{R}^2 : |x-y| \leq r\}$ and the stopping time $T_{\overline{B}(x,r)} := \inf\{t \geq 0 : Y_t \in \overline{B}(x,r)\}$. Use the fact that for any $x \in \mathbb{R}^2$ and $r > 0$, we have $T_{\overline{B}(0,r)} < \infty$ $\mathbb{P}_x$-a.s., and apply the strong Markov property of Brownian motion.
Remark: This exercise shows the *recurrence of Brownian motion in* $\mathbb{R}^2$.

Exercise 6.2

(a) Let $(Z_t)_{t\geq 0}$ be an adapted process with respect to a given filtration $(\mathcal{F}_t)_{t\geq 0}$ such that for every bounded continuous function f, we have

$$E\big[f(Z_t - Z_s)\,\big|\,\mathcal{F}_s\big] = E\big[f(Z_{t-s})\big].$$

Show that Z has stationary independent increments.

(b) Let W be a Brownian motion on $(\Omega, \mathcal{F}, P)$. For every $a \geq 0$, consider the entrance time

$$T_a := \inf\{s \geq 0 : W_s \geq a\}.$$

Show that the process $(T_a)_{a\geq 0}$ has stationary independent increments.

(c) Let $(Z_t)_{t\geq 0}$ have stationary independent increments and start at 0, (W_t) be a Brownian motion independent of (Z_t), and $(T_a)_{a\geq 0}$ as in (b). Show that $(\hat{Z}_t)_{t\geq 0} = (Z_{T_t})_{t\geq 0}$ has stationary independent increments.

Remark: The process $(T_t)_{t\geq 0}$ is called a subordinator.

Exercise 6.3 Let $\phi : \mathbb{R}^k \to \mathbb{R}^k$ be a locally Lipschitz function with linear growth, meaning that

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq C_n|x-y| \quad \text{for some } C_n \geq 0 \text{ and all } x, y : |x|, |y| \leq n, \text{ and} \\ |\phi(x)| &\leq C(1+|x|) \quad \text{for some } C \geq 0. \end{aligned}$$

For each $x \in \mathbb{R}^k$, define $X^x : [0,\infty) \to \mathbb{R}^k$ as the solution to the ODE

$$\begin{cases} \dfrac{dX_t^x}{dt} = \phi(X_t^x), \quad t \geq 0, \\ X_0^x = x. \end{cases} \tag{1}$$

Due the assumptions on ϕ, each ODE has a unique solution X^x (you do not need to prove this).

(a) Define the unique probability measures $\mathbb{P}_x$ on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ such that

$$\mathbb{P}_x[Y_t = X_t^x] = 1$$

for all $x \in \mathbb{R}^k$ and $t \geq 0$. Show that Y is a strong Markov process.

(b) Construct an example where $\phi : \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function and (X^x) is a solution to (1) for each x, but the measures

$$\mathbb{P}_x[Y_t = X_t^x] = 1$$

do not define a Markov process.