Brownian Motion and Stochastic Calculus

Exercise sheet 7

Exercise 7.1 Let $(W_t)_{t\geq 0}$ be a 2-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ started at 0 and $C \neq \emptyset$ an open cone in $\mathbb{R}^2$, i.e. C is an open set and for every $x \in C$, we have $\lambda x \in C$ for all $\lambda > 0$. Note that 0 need not belong to C. Consider the hitting time T_C^W of C, i.e.

$$T_C^W := \inf\big\{t > 0 : W_t \in C\big\}.$$

Show that $T_C^W = 0$ P-a.s.

Exercise 7.2 Let $\Omega = C([0,\infty);\mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ denote the coordinate process. For each $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the unique probability measure on $(\Omega, \mathcal{Y}_\infty^0)$ under which Y is a (d-dimensional) Brownian motion started at x. Moreover, for any open set $A \subseteq \mathbb{R}^d$, we denote by

$$\tau_A := \inf\{t \geq 0 : Y_t \notin A\}$$

the first exit time of the Brownian motion Y from the set A.

Fix an open set $G \subseteq \mathbb{R}^d$ such that $\mathbb{E}_x[\tau_G] < \infty$ for all $x \in G$, a bounded Borel function $g : \mathbb{R}^d \to \mathbb{R}$, and define the function $u : G \to \mathbb{R}$ by

$$u(y) := \mathbb{E}_y\left[\int_0^{\tau_G} g(Y_s)\,ds\right].$$

Moreover, for any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we let $U_\varepsilon(x) := \{y : |y - x| < \varepsilon\}$ denote the open ε-ball around x and set $\sigma_\varepsilon(x) := \tau_{U_\varepsilon(x)}$.

Fix $\varepsilon > 0$ and $x \in G$ such that $U_\varepsilon(x) \subseteq G$. Show that

$$u(x) = \mathbb{E}_x\left[u\left(Y_{\sigma_\varepsilon(x)}\right) + \int_0^{\sigma_\varepsilon(x)} g(Y_s)\,ds\right].$$

Hint: First show that $\tau_G = \tau_G \circ \vartheta_{\sigma_\varepsilon(x)} + \sigma_\varepsilon(x)$. Then compute $u(x)$ by conditioning on $\mathcal{F}_{\sigma_\varepsilon(x)}$ and using the strong Markov property.

Exercise 7.3 Assume we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

(a) Let $M \in \mathcal{M}_{0,\mathrm{loc}}^c$. Prove that $M \in \mathcal{H}_0^{2,c}$ if and only if $E[\langle M\rangle_\infty] < \infty$, and that in this case $\|M\|_{\mathcal{H}^2}^2 = E[\langle M\rangle_\infty]$.

(b) An optional process X is said to be of class (DL) if for all $a > 0$, the family

$$\mathfrak{X}_a := \big\{X_\tau : \tau \text{ stopping time, } \tau \leq a \text{ } P\text{-a.s.}\big\}$$

is uniformly integrable. Show that a local martingale null at 0 is a (true) martingale null at 0 if and only if it is of class (DL).

Remarks:

- As a consequence, we obtain that a local martingale M null at 0 and with integrable supremum, i.e. $M_t^* := \sup_{0\leq s\leq t} |M_s| \in L^1(P)$ for all $t \geq 0$, is a true martingale.

- There exist local martingales null at 0 which are uniformly integrable (i.e. the family $\{M_t : t \geq 0\}$ is uniformly integrable), but are not true martingales.

Exercise 7.4 For any function $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$, we define its total variation (or 1-variation) $|f| : [0, \infty) \to [0, \infty]$ by

$$\begin{aligned} |f|(t) &:= \sup \left\{ \sum_{t_i \in \Pi} \big| f(t_{i+1}) - f(t_i) \big| : \Pi \text{ is a partition of } [0,t] \right\} \\ &= \sup \left\{ \sum_{t_i \in \Pi} \big| f(t_{i+1} \wedge t) - f(t_i \wedge t) \big| : \Pi \text{ is a partition of } [0,\infty) \right\}. \end{aligned}$$

We say that f has finite variation (FV) if $|f|(t) < \infty$ for all $t \geq 0$.

(a) Show that f has finite variation if and only if there exist two non-decreasing functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ with $f_1(0) = f_2(0) = 0$ such that $f = f_1 - f_2$.

If so, find the minimal such functions f_1 and f_2, in the sense that $\tilde{f}_1 \geq f_1$ and $\tilde{f}_2 \geq f_2$ for any other non-decreasing functions $\tilde{f}_1, \tilde{f}_2$ with $\tilde{f}_1(0) = \tilde{f}_2(0) = 0$ such that $f = \tilde{f}_1 - \tilde{f}_2$.

Hint: Start by showing that $|f|(t) - |f|(s) \geq |f(t) - f(s)|$ for $0 \leq s \leq t$.

(b) Show that if f is right-continuous and has finite variation, then $|f|$ is right-continuous.

Using the Carathéodory extension theorem, one can show that for any non-decreasing right-continuous function $\tilde{f}$, there exists a unique positive measure $\mu_{\tilde{f}}$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_{\tilde{f}}((0,t]) = \tilde{f}(t) - \tilde{f}(0)$ for all $t \geq 0$.

(c) Let f be right-continuous with finite variation with $f(0) = 0$ and $g : [0, \infty) \to \mathbb{R}$ such that

$$\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty.$$

Let f_1, f_2 be the minimal functions defined in (a). Show that

$$\int_0^\infty |g(s)| \, d\mu_{f_1}(s) < \infty, \quad \int_0^\infty |g(s)| \, d\mu_{f_2}(s) < \infty,$$

so that

$$\int g(s) \, df(s) := \int g(s) \, d\mu_{f_1}(s) - \int g(s) \, d\mu_{f_2}(s)$$

is well defined.

Remark: If f is of finite variation and right-continuous, a function g is *f-integrable in the Lebesgue–Stieltjes sense* if g satisfies $\int_0^\infty |g(s)| \, \mu_{|f|}(ds) < \infty$. In that case, we define the *Lebesgue–Stieltjes integral* to be $\int g(s) \, df(s)$.