Brownian Motion and Stochastic Calculus

Exercise sheet 8

Exercise 8.1 Let (N_t) be a Poisson process with rate $\lambda > 0$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

(a) Find a martingale M and a predictable finite variation process A both null at 0 such that $N_t = M_t + A_t$ for all $t \geq 0$.

(b) Compute $[M]$ and $\langle M \rangle$ for your choice in (a).

(c) Check by direct calculations that $M^2 - [M]$ is a martingale.

Exercise 8.2 Let $(M_t)_{t\geq 0}$ be a local martingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, i.e. there exists a sequence of stopping times (τ_n) such that $\tau_n \nearrow \infty$ a.s. and each process $M^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is a martingale. Suppose that $\mathbb{F}$ satisfies the usual conditions.

Show that if $M_0 \in L^1$ and $M \geq 0$, i.e. $M_t \geq 0$ P-a.s. for all $t \geq 0$, then M is a supermartingale.

Exercise 8.3

(a) For any $M \in \mathcal{M}^c_{0,\mathrm{loc}}$, define as usual $M^*_t := \sup_{0\leq s\leq t} |M_s|$ for $t \geq 0$. Prove that for any $t \geq 0$ and $C, K > 0$, we have

$$P\big[M^*_t > C\big] \leq \frac{4K}{C^2} + P\big[\langle M \rangle_t > K\big].$$

Hint: Stop $\langle M \rangle$ and use the Markov and Doob inequalities.

Remark: This result allows us to control the running supremum of M in terms of the quadratic variation of M.

(b) Let M be a right-continuous local martingale null at 0. Show that there exists a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ such that M^{τ_n} is a uniformly integrable martingale for each n.

Exercise 8.4

(a) Let $f, g : [0, \infty) \to \mathbb{R}$ be such that g is right-continuous and has finite variation and f is g-integrable in the Lebesgue–Stieltjes sense. Show that the function $h(t) := \int_0^t f(s)\, dg(s)$ is right-continuous. Moreover, show that if g is continuous, then h is continuous.

(b) Let $f : [0, \infty) \to \mathbb{R}$ be a function with finite variation. Show that f has left and right limits, i.e. the limits $f(t-) = \lim_{s\nearrow t} f(s)$ and $f(t+) = \lim_{u\searrow t} f(u)$ exist for $t > 0$ and $t \geq 0$, respectively.

(c) Let $f, g : [0, \infty) \to \mathbb{R}$ be two right-continuous functions of finite variation. Show the integration-by-parts formula, i.e. show that for each $t > 0$,

$$f(t)\, g(t) - f(0)\, g(0) = \int_0^t f(s)\, dg(s) + \int_0^t g(s-)\, df(s) = \int_0^t f(s-)\, dg(s) + \int_0^t g(s)\, df(s).$$

(d) Show also the formula

$$f(t)\, g(t) - f(0)\, g(0) = \int_0^t f(s-)\, dg(s) + \int_0^t g(s-)\, df(s) + \sum_{0<s\leq t} \Delta f(s) \Delta g(s),$$

where $\Delta f(t) = f(t) - f(t-)$ and $\Delta g(t) = g(t) - g(t-)$.

Exercise 8.5 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let Z be a random variable which is symmetric around 0 and not in L^1, that is, $Z \overset{d}{=} -Z$ and $E[Z^+] = E[Z^-] = \infty$. As an example, one can let Z have a Cauchy distribution with density $f_Z(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$. Consider the discrete filtration

$$\mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_1 := \sigma(|Z|), \quad \mathcal{F}_2 := \sigma(Z)$$

and the stochastic process $(X_i)_{i=0,1,2}$ defined by $X_0 = X_1 = 0$ and $X_2 = Z$. Show that X is a local martingale with respect to $\mathbb{F}$, but not a martingale.