Brownian Motion and Stochastic Calculus

Exercise sheet 9

Exercise 9.1 Let W be a Brownian motion with respect to its natural filtration. Show that

$$M_t^{(1)} = e^{t/2}\cos W_t, \quad M_t^{(2)} = tW_t - \int_0^t W_u du, \quad M_t^{(3)} = W_t^3 - 3tW_t$$

are martingales.

Remark: Prove this with a different method than the one used in Exercise **3.3**.

Hint: Recall that for each $t \geq 0$, the running maximum $W_t^* := \sup_{0\leq s\leq t}|W_s|$ has the same distribution as $|W_t|$.

Exercise 9.2 Let $W = (W_t)_{t\geq 0}$ be a 1-dimensional Brownian motion.

(a) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial. Show that the process $H = p(W)$ is in $L^2_{\mathrm{loc}}(W)$, and therefore the stochastic integral $\int p(W)dW$ is well defined. Moreover, show that $\int p(W)dW$ is also a martingale.

(b) For what polynomials $p : \mathbb{R}^2 \to \mathbb{R}$ is the process $X_t = p(W_t, t)$ a martingale? Given a fixed $\lambda \in \mathbb{R}$, for what polynomials $p : \mathbb{R}^2 \to \mathbb{R}$ is the process $Y_t = e^{-\lambda t}p(W_t, t)$ a martingale?

(c) Let W' be another Brownian motion independent of W and ρ a predictable process satisfying $|\rho| \leq 1$. Prove that the process $B = (B_t)_{t\geq 0}$ given by

$$B_t = \int_0^t \rho_s \, dW_s + \int_0^t \sqrt{1-\rho_s^2}\, dW'_s$$

is a Brownian motion. Moreover, compute $\langle B, W\rangle$.

Remark: The pair (W, B) is sometimes called *correlated Brownian motion* with *instantaneous correlation* ρ.

(d) Define $Y_t = (\cos W_t, \sin W_t)^\top$ and $Z_t = (-\sin W_t, \cos W_t)^\top$. Show that Y is not a martingale, but $Z \bullet Y$ is a martingale.

Exercise 9.3 Let $M \in \mathcal{H}_0^{2,c}$. Show that $b\mathcal{E}$ is dense in $L^2(M)$.

Hint: Let $\overline{\Omega} = \Omega \times [0,\infty)$ be equipped with the predictable σ-algebra $\mathcal{P}$. Let $C = E[M_\infty^2]$ and consider the probability measure $P_M = C^{-1}P \otimes [M]$ on $(\overline{\Omega}, \mathcal{P})$. Let $(\Pi_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,\infty)$ with $\lim_{n\to\infty}|\Pi_n| = 0$. Use the martingale convergence theorem on $(\overline{\Omega}, \mathcal{P}, P_M)$ with respect to the discrete filtration $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined by

$$\mathcal{P}_n = \sigma(\{A_i \times (t_i, t_{i+1}] : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}).$$

Exercise 9.4 Let $d \geq 2$, $\Omega = C([0,\infty); \mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ denote the coordinate process. For each $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the unique probability measure on $(\Omega, \mathcal{Y}_\infty^0)$ under which Y is a d-dimensional Brownian motion started at x.

Let $x \in \mathbb{R}^d \setminus \{0\}$ and a, b such that $0 < a < |x| < b$. Consider the stopping times

$$\tau_a := \inf\big\{t \geq 0 : |Y_t| \leq a\big\}, \quad \tau_b := \inf\big\{t \geq 0 : |Y_t| \geq b\big\}.$$

(a) Suppose that $d \geq 3$. Show that $(X_t)_{t\geq 0}$ defined by $X_t := |Y_{\tau_a \wedge t}|^{2-d}$ is a bounded martingale under $\mathbb{P}_x$.

(b) Suppose that $d = 2$. Show that $(X_t)_{t\geq 0}$ defined by $X_t := -\log |Y_{\tau_a \wedge \tau_b \wedge t}|$ is a bounded martingale under $\mathbb{P}_x$.

(c) Let $d \geq 2$. Show that for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\mathbb{P}_x\big[Y_t \neq 0 \text{ for all } t \geq 0\big] = 1.$$

(d) Let $d \geq 3$. Show that for any $x \in \mathbb{R}^d$, we have

$$\mathbb{P}_x\Big[\lim_{t\to\infty} |Y_t| = \infty\Big] = 1.$$

Remark: The result in (d) is known as *transience of Brownian motion in* $\mathbb{R}^d$, *for* $d \geq 3$.