Brownian Motion and Stochastic Calculus

Exercise sheet 0

This exercise sheet is optional and the solutions are not to be submitted. These exercises may be helpful with recalling topics from the "Probability Theory" course, or in (quickly) learning them in preparation for the present course.

Exercise 0.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and $Y_1, Y_2, \dots$ a sequence of independent random variables such that $E[Y_k] = 0$ and $E\big[|Y_k|^\alpha\big] \le \frac{1}{k^\beta}$ for some $\alpha > 1$ and $\beta > 0$.

(a) Show that the process $(S_k)_{k\ge 0}$ defined by $S_k := \sum_{j=1}^k Y_j$ is a martingale with respect to its natural filtration.

(b) Give a sufficient condition on α and β such that (S_k) converges P-almost surely as $k \to \infty$. For which $p \ge 1$ does it follow that (S_k) converges in L^p as $k \to \infty$?

(c) For $M > 0$, let τ_M be the hitting time defined by

$$\tau_M := \inf\{k \ge 0 : |S_k| \ge M\}.$$

Assuming the condition from (b), show that $P[\tau_M < \infty] = O(M^{-\alpha})$ as $M \to \infty$.

Solution 0.1

(a) We observe that the natural filtration of (S_k) is given by $\mathcal{F}_k := \sigma(S_1, \dots, S_k) = \sigma(Y_1, \dots, Y_k)$. Clearly, S is adapted. Using the fact that

$$E\big[|Y_k|\big] \le E\big[|Y_k|^\alpha\big]^{1/\alpha} < \infty$$

by Hölder's inequality and the assumption, we find that the Y_k are integrable and therefore so are the S_k. Finally, by independence it follows that

$$\begin{aligned} E[S_{k+1} - S_k \mid \mathcal{F}_k] &= E\big[Y_{k+1} \mid \sigma(Y_1, \cdots, Y_k)\big] \\ &= E[Y_{k+1}] \\ &= 0 \end{aligned}$$

so that (S_k) is a martingale.

(b) We use Doob's supermartingale convergence theorem. Since (S_k) is a martingale, the P-almost sure convergence follows if

$$\sup_{k\ge 1} E\big[|S_k|\big] < \infty.$$

Note that we have the inequality

$$E\big[|S_k|\big] \le E\big[|S_k|^\alpha\big]^{1/\alpha} \le \sum_{j=1}^k E\big[|Y_j|^\alpha\big]^{1/\alpha} \le \sum_{j=1}^k \frac{1}{j^{\beta/\alpha}} \le \sum_{j=1}^\infty \frac{1}{j^{\beta/\alpha}} < \infty$$

if $\beta/\alpha > 1$, and the last inequality is uniform over k. Therefore, (S_k) converges P-almost surely if $\beta > \alpha$.

For the L^p-convergence, we can only ensure that $S_k \in L^p$ if $p \le \alpha$, so that

$$E\big[|S_k|^p\big]^{1/p} \le E\big[|S_k|^\alpha\big]^{1/\alpha} \le \sum_{j=1}^{k} E\big[|Y_j|^\alpha\big]^{1/\alpha} < \infty.$$

We have that (S_k) is bounded in L^α, since

$$\sup_{k\ge 1} E\big[|S_k|^\alpha\big]^{1/\alpha} \le \sum_{j=1}^{\infty} \frac{1}{j^{\beta/\alpha}} < \infty.$$

It follows from a corollary to the martingale convergence theorem[1] that (S_k) converges in L^1 and L^α to S_∞. This also gives convergence in L^p for any $p \in [1, \alpha]$.

(c) We have that

$$P[\tau_M < \infty] = P\left[\sup_{j\ge 1} |S_j| \ge M\right] = \lim_{k\to\infty} P\left[\sup_{1\le j\le k} |S_j| \ge M\right].$$

Note that $(|S_j|^\alpha)$ is a submartingale by Jensen's inequality. We can use Doob's inequality to bound

$$P\left[\sup_{1\le j\le k} |S_j| \ge M\right] = P\left[\sup_{1\le j\le k} |S_j|^\alpha \ge M^\alpha\right] \le \frac{1}{M^\alpha} E\left[|S_k|^\alpha\right] \le \frac{1}{M^\alpha}\left(\sum_{j=1}^{\infty} \frac{1}{j^{\beta/\alpha}}\right)^\alpha.$$

Since this bound is uniform over k, we find that

$$P[\tau_M < \infty] \le \min\left(1, \frac{1}{M^\alpha}\left(\sum_{j=1}^{\infty} \frac{1}{j^{\beta/\alpha}}\right)^\alpha\right) = O(M^{-\alpha})$$

as $M \to \infty$.

[1] **Corollary.** Let $p > 1$ and let $(M_k)_{k\in\mathbb{N}}$ be a martingale such that $\sup_k E[|M_k|^p] < \infty$. Then, (M_k) converges a.s. and in L^p to $M_\infty \in L^p$.
Proof. The bound in L^p implies that $\{M_k : k \in \mathbb{N}\}$ is uniformly integrable, therefore (M_k) converges a.s. and in L^1 to $M_\infty \in L^1$, by the martingale convergence theorem. By Doob's L^p inequality, it holds that for each $n \in \mathbb{N}$,

$$E\left[\sup_{1\le k\le n} |M_k|^p\right] \le \left(\frac{p}{p-1}\right)^p E[|M_k|^p] \le \left(\frac{p}{p-1}\right)^p \sup_n E[|M_n|^p],$$

and by the monotone convergence theorem we obtain that

$$E\left[\sup_k |M_k|^p\right] \le \left(\frac{p}{p-1}\right)^p \sup_n E[|M_n|^p].$$

It then follows by Fatou's lemma that $M_\infty \in L^p$ and by the dominated convergence theorem (using $2^p \sup_k |M_k|^p$ as the bound for $|M_k - M_\infty|^p$) that $M_k \to M_\infty$ in L^p.

Exercise 0.2 Let X be a real-valued random variable with standard normal distribution as law and Y a random variable independent of X with law defined by

$$P[Y=1]=p \quad \text{and} \quad P[Y=-1]=1-p, \qquad (0 \le p \le 1).$$

We define $Z := XY$.

(a) What is the law of Z? Is the vector (X,Z) a Gaussian vector?

(b) Calculate $\mathrm{Cov}(X,Z)$. For which $p \in [0,1]$ are the random variables X and Z uncorrelated, i.e. $\mathrm{Cov}(X,Z)=0$?

(c) Show that for each $p \in [0,1]$, the random variables X and Z are *not* independent.

Solution 0.2

(a) We show that $Z \sim \mathcal{N}(0,1)$ by calculating its characteristic function. Using the independence of X and Y and that X and $-X \sim \mathcal{N}(0,1)$, we get for each $t \in \mathbb{R}$ that

$$\begin{aligned}\varphi_Z(t) := E\big[e^{itZ}\big] &= E\big[e^{itX}\, 1_{\{Y=1\}}\big] + E\big[e^{-itX}\, 1_{\{Y=-1\}}\big] \\ &= E\big[e^{itX}\big]\ P\big[Y=1\big] + E\big[e^{-itX}\big]\ P\big[Y=-1\big] \\ &= e^{-\frac{1}{2}t^2}.\end{aligned}$$

To check whether (X,Z) is a Gaussian vector, we need to check if for all $\lambda_1, \lambda_2 \in \mathbb{R}$, the random variable $\lambda_1 X + \lambda_2 Z$ is normally distributed. Fix any $\lambda_1, \lambda_2 \in \mathbb{R}$.
For $p \in \{0,1\}$ we see that

$$\lambda_1 X + \lambda_2 Z = cX$$

for some $c \in \{\lambda_1+\lambda_2, \lambda_1-\lambda_2\}$. Therefore, as $X \sim \mathcal{N}(0,1)$ we get that $\lambda_1 X + \lambda_2 Z \sim \mathcal{N}(0,c^2)$ and thus (X,Z) is a Gaussian vector.
Now, let $p \in [0,1] \setminus \{0,1\}$. Assume by contradiction that (X,Z) is a Gaussian vector. Then, $X+Z$ is normally distributed. But then $P[X+Z=0] \in \{0,1\}$ as $X+Z$ is normally distributed, which contradicts the fact that

$$P\big[X+Z=0\big] = P\big[Y=-1 \text{ or } X=0\big] = 1-p \notin \{0,1\}.$$

We conclude that

$$(X,Z) \text{ is a Gaussian vector} \iff p \in \{0,1\}.$$

(b) Using that $X \sim \mathcal{N}(0,1)$, the independence of X and Y and that $E[Y]=2p-1$, we get

$$\begin{aligned}\mathrm{Cov}\big(X,Z\big) &= E\big[X^2\,Y\big] - E\big[X\big]\ E\big[X\,Y\big] \\ &= E\big[X^2\big]\ E\big[Y\big] \\ &= \mathrm{Var}\big(X\big)\ E\big[Y\big] \\ &= 2p-1.\end{aligned}$$

Therefore,

$$\mathrm{Cov}\big(X,Z\big) = 0 \iff p = 1/2.$$

(c) Assume by contradiction that X and Z are independent. Then, as $Z \sim \mathcal{N}(0,1)$,

$$0 = P\Big[|Z|>1 \,\Big|\, |X| \le 1\Big] = P\Big[|Z|>1\Big] \neq 0$$

which gives a contradiction.
Alternative proof: For $p \in (0,1)$, if X and Z were independent, (X,Z) would be a Gaussian vector, since X and Z are normally distributed by (a). This contradicts the second part of (a). For $p \in \{0,1\}$, it is clear that we do not have independence, since in that case

$$X = Z \text{ a.s.} \quad \text{or} \quad X = -Z \text{ a.s.}$$

Exercise 0.3 We consider several examples of weak convergence. Results related with characteristic functions may be helpful with the proofs.

(a) Construct a sequence of rescaled binomial random variables X_n and a standard normal random variable X such that $X_n \Rightarrow X$ as $n \to \infty$.

(b) Construct a sequence of rescaled binomial random variables X_n and a Poisson random variable X such that $X_n \Rightarrow X$ as $n \to \infty$.

(c) Construct a sequence of rescaled geometric random variables X_n and an exponential random variable X such that $X_n \Rightarrow X$ as $n \to \infty$.

(d) Let X be a real-valued random variable with distribution function F. Construct a sequence of random variables X_n such that $X_n \Rightarrow X$ as $n \to \infty$ and each X_n has a continuous density function f_n.

Solution 0.3

(a) Let $X_n = \frac{Y_n - np}{\sqrt{np(1-p)}}$, where $Y_n \sim B(n,p)$ for $p \in (0,1)$. Then $X_n \Rightarrow \mathcal{N}(0,1)$. This can be shown with the central limit theorem: if $Z_1, Z_2, \ldots$ are independent Bernoulli random variables with parameter p, then the equality in law $Y_n \stackrel{d}{=} \tilde{Y}_n = \sum_{j=1}^n Z_j$ holds. It follows by the CLT that $X_n \stackrel{d}{=} \frac{\tilde{Y}_n - np}{\sqrt{np(1-p)}} \Rightarrow \mathcal{N}(0,1)$.

(b) Let $X_n \sim B(n, \lambda/n)$ for some fixed $\lambda \in \mathbb{R}$ and $n > \lambda$. The characteristic function is

$$\psi_{X_n}(t) = \left(1 + \frac{\lambda}{n}\left(e^{it} - 1\right)\right)^n.$$

Taking the limit as $n \to \infty$, we obtain that for each $t \in \mathbb{R}$,

$$\psi_{X_n}(t) \to \psi(t) = \exp\Big(\lambda\big(e^{it} - 1\big)\Big).$$

This is the characteristic function of a Poisson random variable $X \sim \mathrm{Poi}(\lambda)$; therefore $X_n \Rightarrow X$.

(c) Let $X_n = \frac{1}{n} Y_n$ where $Y_n \sim \mathrm{Geom}(p/n)$, for $p \in (0,1)$. The characteristic function is

$$\psi_{X_n}(t) = \frac{\frac{p}{n} e^{it/n}}{1 - \left(1 - \frac{p}{n} e^{it/n}\right)} = \frac{p}{ne^{-it/n} - n + p}.$$

Noting that $e^{-it/n} = 1 - it/n + O(1/n^2)$, we obtain that for each $t \in \mathbb{R}$,

$$\psi_{X_n}(t) \to \psi(t) = \frac{p}{p - it}.$$

This is the characteristic function of an exponential random variable $X \sim \mathrm{Exp}(p)$; therefore $X_n \Rightarrow X$.

(d) One idea is to perturb X by an independent random variable $Y_n \sim \mathcal{N}\left(0, \frac{1}{n^2}\right)$. This leads us to construct random variables X_n with cumulative distribution function

$$F_n(x) = \int_{-\infty}^{+\infty} \Phi\big(n(x-y)\big) dF(y),$$

where Φ is the distribution function of a standard normal random variable. We note that F_n is increasing for each n, with $F_n(-\infty) = 0$ and $F_n(+\infty) = 1$. Moreover, F_n is continuous and even differentiable with density

$$f_n(x) = \int_{-\infty}^{+\infty} \exp\left(-\frac{n^2(x-y)^2}{2}\right) dF(y).$$

Finally, we note using the dominated convergence theorem that

$$\lim_{n\to\infty} F_n(x) = \int_{-\infty}^{+\infty} \left(\mathbb{1}_{\{y<x\}} + \frac{1}{2}\mathbb{1}_{\{y=x\}}\right) dF(y) = F(x)$$

at points of continuity of F, therefore $X_n \Rightarrow X$.