Brownian Motion and Stochastic Calculus

Exercise sheet 1

Exercise 1.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X, Y, Z : \Omega \to \mathbb{R}$ be random variables and suppose that Z is $\sigma(X, Y)$-measurable. Use the monotone class theorem to show that there exists a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $Z = f(X, Y)$.

Hint: It may be helpful to start by assuming that Z is bounded.

Solution 1.1 Define $\mathcal{H}$ as the set of bounded random variables Z such that $Z = f(X, Y)$ for some measurable function $f : \mathbb{R}^2 \to \mathbb{R}$. Moreover, define

$$\mathcal{M} = \{\mathbb{1}_{\{X \in A\}} \mathbb{1}_{\{Y \in B\}} : A, B \in \mathcal{B}(\mathbb{R})\}.$$

It is clear that $\mathcal{M}$ is closed under multiplication with

$$\sigma(\mathcal{M}) = \sigma\Big(\big\{\{X \in A\} \cap \{Y \in B\} : A, B \in \mathcal{B}(\mathbb{R})\big\}\Big) = \sigma(X, Y).$$

Note that $\mathcal{H}$ contains the constant 1 and $\mathcal{M}$.

We show that $\mathcal{H}$ is closed under monotone bounded convergence. Suppose that (Z_k) is a sequence of random variables such that $0 \le Z_1 \le Z_2 \le \cdots$ and $Z_k = f_k(X, Y)$, and suppose that $Z = \sup_k Z_k$ is bounded by $C > 0$. We have that $Z = f(X, Y)$ where $f(x, y) = \sup_k f_k(x, y) \wedge C$ is a measurable function, since $Z = \sup_k f_k(X, Y) = f(X, Y)$ using the bound on Z. Therefore, by the monotone class theorem, $\mathcal{H}$ contains all bounded random variables that are $\sigma(X, Y)$-measurable.

Consider now a general $\sigma(X, Y)$-measurable random variable Z. For each $n \in \mathbb{N}$, define $g^n, g^\infty : [0, \infty] \to \mathbb{R}$ by $g^n(x) = x\mathbb{1}_{x \le n}$ and $g^\infty(x) = x\mathbb{1}_{x < \infty}$. We can write

$$g^n(Z^+) = f_n^{(+)}(X, Y), \qquad g^n(Z^-) = f_n^{(-)}(X, Y)$$

for non-negative measurable functions $f_n^{(+)}$ and $f_n^{(-)}$. Non-negativity can be ensured by replacing $f_n^{(+)}$ with $f_n^{(+)} \vee 0$, etc. Likewise, we can ensure that $\left(f_n^{(+)}\right)$ and $\left(f_n^{(-)}\right)$ are increasing in n, by replacing $f_{n+1}^{(+)}$ with $f_{n+1}^{(+)} \vee f_n^{(+)}$, and analogously for $f_{n+1}^{(-)}$. This is justified since $f_n^{(+)}(X, Y) = g^n(Z^+) \le g^{n+1}(Z^+) = f_{n+1}^{(+)}(X, Y)$.

Define $f^{(+)}(x, y) = g^\infty\left(\sup_n f_n^{(+)}(x, y)\right)$ and $f^{(-)}(x, y) = g^\infty\left(\sup_n f_n^{(-)}(x, y)\right)$. We obtain that

$$\begin{aligned} Z &= \sup_{n \in \mathbb{N}} (Z\mathbb{1}_{0 \le Z \le n}) - \sup_{n \in \mathbb{N}} (-Z\mathbb{1}_{-n \le Z \le 0}) \\ &= g^\infty\left(\sup_{n \in \mathbb{N}} f_n^{(+)}(X, Y)\right) - g^\infty\left(\sup_{n \in \mathbb{N}} f_n^{(-)}(X, Y)\right) \\ &= f(X, Y) \end{aligned}$$

since $|Z| < \infty$, and where $f(x, y) = g^\infty(\sup_{n \in \mathbb{N}} f_n^{(+)}(x, y)) - g^\infty(\sup_{n \in \mathbb{N}} f_n^{(-)}(x, y))$.

Exercise 1.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ are two stochastic processes on $(\Omega, \mathcal{F}, P)$. Two processes Z and Z' on $(\Omega, \mathcal{F}, P)$ are said to be *modifications* of each other if $P[Z_t = Z'_t] = 1,\ \forall\ t \geq 0$, while Z and Z' are *indistinguishable* if $P[Z_t = Z'_t,\ \forall\ t \geq 0] = 1$.

(a) Assume that X and Y are both right-continuous or both left-continuous. Show that the processes are modifications of each other if and only if they are indistinguishable.

Remark: A stochastic process is said to *have the path property* $\mathcal{P}$ ($\mathcal{P}$ can be continuity, right-continuity, differentiability, ...) if the property $\mathcal{P}$ holds for P-almost every path.

(b) Give an example showing that one of the implications of part (a) does not hold for general X, Y.

Solution 1.2

(a) We show that if X is a modification of Y, then they are indistinguishable, since the converse is obvious. Assume that X and Y are right-continuous; the proof for the left-continuous case is analogous.

For each $t \geq 0$, we define the nullset $N_t := \{\omega : X_t(\omega) \neq Y_t(\omega)\}$. We consider $N := \cup_{t\in\mathbb{Q}_+} N_t$, which remains a nullset as a countable union of null sets. Finally, we introduce the nullset $A_Z := \{\omega : Z_\cdot(\omega) \text{ not right-continuous}\}$ for $Z = X, Y$ and we define $M := A_X \cup A_Y \cup N$, which is still a nullset.

For any given $\omega \in M^c$, it remains to check that $X_t(\omega) = Y_t(\omega),\ \forall\ t \geq 0$. By definition of M, it holds that $X_t(\omega) = Y_t(\omega),\ \forall\ t \in \mathbb{Q}_+$. Now, take any $t \geq 0$ and let (t_n) be a sequence in $\mathbb{Q}_+$ with $t_n \downarrow t$. The right-continuity of the paths $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ then implies that $X_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_t(\omega)$.

(b) Take $\Omega = [0, \infty)$, $\mathcal{F} = \mathcal{B}([0, \infty))$ the Borel σ-algebra, and P a probability measure with $P[\{\omega\}] = 0,\ \forall\ \omega \in \Omega$ (for instance, the exponential distribution).

Set $X \equiv 0$ and $Y_t(\omega) = \begin{cases} 1, & t = \omega, \\ 0, & \text{else.} \end{cases}$

Then $P[X_t = Y_t] = 1,\ \forall\ t \geq 0$, since single points have no mass, but $\{X_t = Y_t,\ \forall\ t \geq 0\} = \emptyset$. Note that all sample paths of X are continuous, while all sample paths of Y are discontinuous at $t = \omega$.

Exercise 1.3 Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The aim of this exercise is to show the following chain of implications:

$$X \text{ optional} \Rightarrow X \text{ progressively measurable} \Rightarrow X \text{ product-measurable and adapted.}$$

(a) Show that every progressively measurable process is product-measurable and adapted.

(b) Assume that X is adapted and *every* path of X is right-continuous. Show that X is progressively measurable.
Remark: The same conclusion holds true if every path of X is left-continuous.
Hint: For fixed $t \geq 0$, consider an approximating sequence of processes Y^n on $\Omega \times [0, t]$ given by $Y_0^n = X_0$ and $Y_u^n = \sum_{k=0}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$.

(c) Recall that the optional σ-field $\mathcal{O}$ is generated by the class $\overline{\mathcal{M}}$ of all adapted processes whose paths are all RCLL. Show that $\mathcal{O}$ is also generated by the subclass $\mathcal{M}$ of all *bounded* processes in $\overline{\mathcal{M}}$.

(d) Use the monotone class theorem to show that every optional process is progressively measurable.

Solution 1.3

(a) Let X be progressively measurable. Then $X|_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}[0, t]$-measurable for every $t \geq 0$. For any $t \geq 0$, we see that $X_t = X \circ i_t$, where $i_t : (\Omega, \mathcal{F}_t) \to (\Omega \times [0, t], \mathcal{F}_t \otimes \mathcal{B}[0, t])$, $\omega \mapsto (\omega, t)$ is measurable. Therefore, X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$. Moreover, the processes X^n defined by $X_u^n := X|_{\Omega\times[0,n]} 1_{[0,n]}(u)$, $u \geq 0$, are $\mathcal{F} \otimes \mathcal{B}[0, \infty)$-measurable. Since $X^n \to X$ pointwise (in (t, ω)) as $n \to \infty$, also X is $\mathcal{F} \otimes \mathcal{B}[0, \infty)$-measurable.

(b) Fix a $t \geq 0$ and consider the sequence of processes Y^n on $\Omega \times [0, t]$ given by $Y_0^n = X_0$ and $Y_u^n = \sum_{k=1}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}$ for $u \in (0, t]$. By construction, each Y^n is $\mathcal{F}_t \otimes \mathcal{B}[0, t]$-measurable. Since $Y^n \to X|_{\Omega\times[0,t]}$ pointwise as $n \to \infty$ due to right-continuity, the result follows.

(c) Let X be adapted, with all paths being RCLL. Consider the processes $X^n := (X \wedge n) \vee (-n)$. Clearly, each X^n is bounded and RCLL. Thus, each X^n is $\sigma(\mathcal{M})$-measurable. As the pointwise limit of the X^n, also X is $\sigma(\mathcal{M})$-measurable. It follows that $\mathcal{O} \subseteq \sigma(\mathcal{M})$. The converse inclusion is trivial.

(d) If a process X is optional, then $X^n := X\, 1_{\{|X|\leq n\}}$ is also optional and of course $X^n \to X$; so if each X^n is progressively measurable, then so is X, and hence we can assume without loss of generality that X is bounded.
Let $\mathcal{H}$ denote the real vector space of bounded, progressively measurable processes. By part b), $\mathcal{H}$ contains $\mathcal{M}$. Clearly, $\mathcal{H}$ contains the constant process 1 and is closed under monotone bounded convergence. Also, $\mathcal{M}$ is closed under multiplication. The monotone class theorem yields that every bounded $\sigma(\mathcal{M})$-measurable process is contained in $\mathcal{H}$. Due to c), we conclude that every bounded optional process is progressively measurable.

Exercise 1.4

(a) Let $(\Omega, \mathcal{F}, P)$ be a probability space and B a Brownian motion on $[0,1]$. Let $k \in \mathbb{N}$ and

$$0 = s_1 < t_1 < s_2 < t_2 < \ldots < t_k < s_{k+1} = 1.$$

Find the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditional on $(B_{s_1}, \ldots, B_{s_{k+1}})$.

(b) Let $\mathcal{D} := \{a2^{-m} : m \in \mathbb{N}, a \in \{0, 1, \ldots, 2^m\}\}$. Let $Z_1, Z_2, \ldots$ be an infinite sequence of i.i.d. standard normal random variables. Construct in terms of the Z_j a stochastic process $(W_t)_{t \in \mathcal{D}}$ such that the law of W is equal to the law of $(B_t)_{t \in \mathcal{D}}$.

Solution 1.4

(a) Note that $(B_{s_1}, B_{t_1}, \ldots, B_{t_k}, B_{s_{k+1}})$ is a Gaussian vector. We claim that for each k,

$$\Delta_k := B_{t_k} - \frac{t_k - s_k}{s_{k+1} - s_k} B_{s_{k+1}} - \frac{s_{k+1} - t_k}{s_{k+1} - s_k} B_{s_k}$$

is normally distributed with $\Delta_k \sim \mathcal{N}\left(0, \frac{(s_{k+1} - t_k)(t_k - s_k)}{s_{k+1} - s_k}\right)$, and moreover Δ_k is independent of $(B_{s_1}, \ldots, B_{s_{k+1}})$.

The first claim follows from the Gaussian distribution and rewriting

$$\Delta_k := -\frac{t_k - s_k}{s_{k+1} - s_k}(B_{s_{k+1}} - B_{t_k}) + \frac{s_{k+1} - t_k}{s_{k+1} - s_k}(B_{t_k} - B_{s_k})$$

where the two increments are independent, from which we get the variance. For the second claim, due to the Gaussian distribution, it is enough to show that Δ_k is uncorrelated with $B_{s_{j+1}} - B_{s_j}$ for each j. This is clear for any $j \neq k$, while at k we have that

$$E[\Delta_k (B_{s_{k+1}} - B_{s_k})] = -\frac{t_k - s_k}{s_{k+1} - s_k}(s_{k+1} - t_k) + \frac{s_{k+1} - t_k}{s_{k+1} - s_k}(t_k - s_k) = 0.$$

Therefore, we conclude that the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditional on $(B_{s_1}, \ldots, B_{s_{k+1}})$ is the Gaussian law $\mathcal{N}(\mu, \Sigma)$, where

$$\mu_k = \frac{t_k - s_k}{s_{k+1} - s_k} B_{s_{k+1}} + \frac{s_{k+1} - t_k}{s_{k+1} - s_k} B_{s_k}$$

and the matrix Σ is diagonal with $\Sigma_{kk} = \frac{(s_{k+1} - t_k)(t_k - s_k)}{s_{k+1} - s_k}$.

(b) Let $\mathcal{D}^n := \{a2^{-m} : m \in \{1, \ldots, n\}, a \in \{0, 1, \ldots, 2^m\}\}$. We construct W recursively on each $\mathcal{D}^n$, so that finally we obtain W on $\mathcal{D}$. The first step is to define $W_1 = Z_1$, so that clearly $W \overset{d}{=} B$ on $\{0, 1\}$. If we have defined W on $\mathcal{D}^n$ in terms of $Z_1, Z_2, \ldots, Z_{2^{n-1}}$, we extend it to $\mathcal{D}^{n+1}$ by

$$W_{(2j-1)2^{-(m+1)}} = \frac{1}{2} W_{(j-1)2^{-m}} + \frac{1}{2} W_{j2^{-m}} + 2^{-\frac{n}{2}-1} Z_{2^n + j}$$

for $j = 1, \ldots, 2^n$. By induction, assume that $W \overset{d}{=} B$ on $\mathcal{D}^n$. We also obtain from this construction that the law of $W|_{\mathcal{D}_{n+1}}$ conditional on $W|_{\mathcal{D}_n}$ is equal to the law of $B|_{\mathcal{D}_{n+1}}$ conditional on $B|_{\mathcal{D}_n}$, by (a). Therefore, the inductive step is valid, and we finally obtain that the law of W is equal to the law of $B|_{\mathcal{D}}$ by the Ionescu-Tulcea theorem.