Brownian Motion and Stochastic Calculus

Exercise sheet 10

Exercise 10.1 Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space satisfying the usual conditions.

(a) Let $W, \tilde{W}$ be two Brownian motions with respect to P and $(\mathcal{F}_t)_{t\geq 0}$. Show that we have $d\langle W, \tilde{W}\rangle_t = \rho_t\,dt$ for some predictable process ρ taking values in $[-1, 1]$.

Hint: Use the Kunita–Watanabe decomposition.

(b) The filtration $\mathbb{F}$ is called *P-continuous* if all local $(P, \mathbb{F})$-martingales are continuous. Show that $\mathbb{F}$ is P-continuous if and only if $\mathbb{F}$ is Q-continuous for all $Q \approx P$.

(c) Suppose that $\mathbb{F}$ is P-continuous and let $S = (S_t)_{t\geq 0}$ be a local $(Q, \mathbb{F})$-martingale for some $Q \approx P$. Show that S is a continuous P-semimartingale of the form

$$S = S_0 + M + \int \alpha\, d\langle M\rangle \tag{1}$$

for some $M \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$ and some $\alpha \in L^2_{\mathrm{loc}}(M)$.

Hint: Use Girsanov's theorem to find a semimartingale decomposition for S under P. Then use the Kunita–Watanabe decomposition under P to describe its finite variation part.

Remark: If S has the form (1), one says that it satisfies the *structure condition SC*. This is a useful concept in mathematical finance.

Solution 10.1

(a) Using the Kunita–Watanabe decomposition, we can write $W = \rho \bullet \tilde{W} + N$ for some predictable integrand $\rho \in L^2_{\mathrm{loc}}(\tilde{W})$ and some local martingale $N \in \mathcal{M}^c_{0,\mathrm{loc}}$ strongly orthogonal to $\tilde{W}$. Then, by orthogonality and associativity of the stochastic integral,

$$\langle W, \tilde{W}\rangle_t = \langle \rho \bullet \tilde{W} + N, \tilde{W}\rangle_t = \int_0^t \rho_s d\langle \tilde{W}\rangle_s + \langle \tilde{W}, N\rangle_t = \int_0^t \rho_s ds, \quad t \geq 0.$$

Moreover, because $\langle \tilde{W}, N\rangle \equiv 0$,

$$\int_0^t 1 ds = t = \langle W\rangle_t = \langle \rho \bullet \tilde{W} + N\rangle_t = \int_0^t \rho_s^2\, ds + \langle N\rangle_t, \quad t \geq 0.$$

Hence $\int_0^\cdot (1 - \rho_s^2)\, ds = \langle N\rangle$ is an increasing process. It follows that $\rho^2 \leq 1$ $dt \otimes P$-a.e.

(b) We show the implication "$\Rightarrow$", since "$\Leftarrow$" is trivial. Fix $Q \approx P$ and let $(Z^Q_t)_{t\geq 0}$ be the density process of Q with respect to P. Since Z^Q is a $(P, \mathbb{F})$-martingale, Z^Q is continuous. Note that $Z^Q_t > 0$ for all $t \geq 0$ a.s., since $Q \approx P$. Therefore, $1/Z^Q$ is also continuous.

Let X be a local $(Q, \mathbb{F})$-martingale Then $Z^Q X$ is a local $(P, \mathbb{F})$-martingale and thus continuous P-a.s. Therefore, $X = \frac{1}{Z^Q}(Z^Q X)$ is continuous P-a.s. As $Q \approx P$, we have that X is also continuous Q-a.s. Since X is an arbitrary local Q-martingale, we find that $\mathbb{F}$ is Q-continuous for any $Q \approx P$.

(c) Let Z^P be the density process of P with respect to Q. Note that $Z_0^P = 1$, and moreover Z^P is strictly positive and continuous by (b). Therefore, we can write $Z^P = \mathcal{E}(L)$ for $L \in \mathcal{M}^c_{0,\mathrm{loc}}(Q)$ defined by $L = \frac{1}{Z^P} \bullet Z_P$.

By Girsanov's theorem, since S is a local Q-martingale, we obtain the local P-martingale

$$M := S - S_0 - \langle L, S - S_0 \rangle \in \mathcal{M}^c_{0,\mathrm{loc}}(P).$$

Rewriting, we have the P-semimartingale decomposition

$$S = S_0 + M + \langle L, S - S_0 \rangle,$$

and it only remains to show that $\langle L, S - S_0 \rangle = \int \alpha \, d\langle M \rangle$ for some $\alpha \in L^2_{\mathrm{loc}}(M)$.

Since $L \in \mathcal{M}^c_{0,\mathrm{loc}}(Q)$, by Girsanov's theorem we have that $\tilde{L} := L - \langle L \rangle \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$. Applying the Kunita–Watanabe decomposition to $\tilde{L}$ with respect to M, we obtain that $\tilde{L} = \int \alpha \, dM + N$ for some $\alpha \in L^2_{\mathrm{loc}}(M)$ and some $N \in \mathcal{M}^c_{0,\mathrm{loc}}(P)$ such that $N \perp M$. Since $S - S_0 - M$ and $L - \tilde{L}$ are continuous finite variation processes, their quadratic variation is 0. Therefore,

$$\langle L, S - S_0 \rangle = \langle \tilde{L}, M \rangle = \left\langle \int \alpha \, dM + N, M \right\rangle = \int \alpha \, d\langle M \rangle,$$

as we wanted.

Exercise 10.2 Let $B = (B^1, B^2, B^3)$ be a Brownian motion in $\mathbb{R}^3$ and $Z = (Z^1, Z^2, Z^3)$ a standard normal random variable. Define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{|Z + B_t|}.$$

Note that $P[B_t \neq x, \forall t \geq 0] = 1$ for any $x \in \mathbb{R}^3 \setminus \{0\}$; see Exercise **9.4**.

(a) Show that $P[B_t \neq -Z, \forall t \geq 0] = 1$, so that M is a.s. well defined.

(b) Show that $|Z + B_t|^2 \sim \text{Gamma}(\frac{3}{2}, \frac{1}{2(t+1)})$ for each $t > 0$, i.e., its density is given by

$$f_t(y) = \frac{(2(t+1))^{-3/2} y^{1/2}}{\Gamma(3/2)} \exp\left(-\frac{y}{2(t+1)}\right), \quad y \geq 0.$$

(c) Show that M is a continuous local martingale. Moreover, show that M is bounded in L^2, i.e., $\sup_{t\geq 0} E[|M_t|^2] < \infty$.

(d) Show that M is a *strict local martingale*, i.e., M is not a martingale.

Hint: Show that $E[M_t] \to 0$ as $t \to \infty$.

Remark: This is the standard example of a local martingale which is not a (true) martingale. It also shows that even boundedness in L^2 (which implies uniform integrability) does not guarantee the martingale property.

Solution 10.2

(a) By independence, we have that

$$P[B_t \neq -Z, \forall t \geq 0] = E\big[P[B_t \neq -x, \forall t \geq 0]|_{x=Z}\big] \geq E[\mathbb{1}_{\{Z\neq 0\}}] = 1,$$

since Z has a Gaussian distribution, so that $P[Z = 0] = 0$.

(b) We first find the distribution $\tilde{f}_t$ of $|Z^1 + B_t^1|^2$. Note that $Z^1 + B_t^1 \sim \mathcal{N}(0, t+1)$. Thus, for $y \geq 0$, we have that

$$\begin{aligned} P[|Z^1 + B_t^1|^2 \leq y] &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{z^2}{2(t+1)}} dz \\ &= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{z^2}{2(t+1)}} dz. \end{aligned}$$

Changing variables to $u = z^2$, we find that

$$P[|B_t^1|^2 \leq y] = 2\int_0^{y} \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{u}{2(t+1)}} \frac{1}{2\sqrt{z}} dz.$$

Differentiating in y, we obtain that

$$\tilde{f}_t(y) = \frac{z^{-1/2}}{(2(t+1))^{1/2}\sqrt{\pi}} e^{-\frac{y}{2(t+1)}} = \frac{z^{-1/2}}{(2(t+1))^{1/2}\Gamma(1/2)} e^{-\frac{y}{2(t+1)}}.$$

Therefore, $|Z^1 + B_t^1|^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2(t+1)})$. By properties of the Gamma distribution, if $Y_1, \dots, Y_n \sim \text{Gamma}(\alpha, \beta)$ are i.i.d. random variables, then $Y_1 + \dots + Y_n \sim \text{Gamma}(n\alpha, \beta)$. Since $Z^1 + B^1, Z^2 + B^2, Z^3 + B^3$ are i.i.d., we have that

$$|Z + B_t|^2 = |Z^1 + B_t^1|^2 + |Z^2 + B_t^2|^2 + |Z^3 + B_t^3|^2 \sim \text{Gamma}\left(\frac{3}{2}, \frac{1}{2(t+1)}\right).$$

(c) Let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}_+$ be defined by $f(y) = \frac{1}{|y|}$. In Exercise **9.4**, we already showed that

$$M_t = f(x + |B_t|) = M_0 + \int_0^t \nabla f(B_s) dB_s,$$

since f is C^2 and harmonic in its domain. In particular, since $\nabla f(B_s)$ is continuous, it is locally bounded so that M is a local martingale.

To show that (M_t) is bounded in L^2, note that by (a),

$$\begin{aligned}
E[M_t^2] &= E\left[\frac{1}{|Z + B_t|^2}\right] \\
&= \int_0^\infty \frac{1}{y} \frac{(2(t+1))^{-3/2} y^{1/2}}{\Gamma(3/2)} \exp\left(-\frac{y}{2(t+1)}\right) dy \\
&= \frac{\Gamma(1/2)}{\Gamma(3/2)} \frac{1}{2(t+1)} \int_0^\infty \frac{(2(t+1))^{-1/2} y^{-1/2}}{\Gamma(1/2)} \exp\left(-\frac{y}{2(t+1)}\right) dy \\
&= \frac{2\Gamma(1/2)}{\Gamma(1/2)} \frac{1}{2(t+1)} \cdot 1 \\
&= \frac{1}{t+1}, \quad t > 0,
\end{aligned}$$

as $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$ and since we integrate the density of a Gamma$(\frac{1}{2}, \frac{1}{2(t+1)})$ distribution. Therefore, $\sup_{t\geq 0} E[M_t^2] = 1 < \infty$.

(d) For $t > 0$,

$$\begin{aligned}
E[M_t] &= \int_0^\infty \frac{1}{\sqrt{y}} \frac{(2(t+1))^{-3/2} y^{1/2}}{\Gamma(3/2)} \exp\left(-\frac{y}{2(t+1)}\right) dy \\
&= \frac{1}{\Gamma(3/2)\sqrt{2(t+1)}} \int_0^\infty (2(t+1))^{-1} \exp\left(-\frac{y}{2(t+1)}\right) dy \\
&= \frac{\sqrt{2}}{\sqrt{\pi t}} \to 0
\end{aligned}$$

as $t \to \infty$. Since $E[M_0] = \frac{1}{|x|} > 0$, M cannot be a martingale.

Exercise 10.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the raw filtration generated by W. Moreover, fix $T > 0$, $a < b$, and let $F := \mathbb{1}_{\{a\leq W_T\leq b\}}$. The goal of this exercise is to find explicitly the integrand $H \in L^2_{\text{loc}}(W)$ in the Itô representation

$$F = E[F] + \int_0^\infty H_s \, dW_s. \qquad (\star)$$

(a) Show that the martingale $M = (M_t)_{t\geq 0}$ given by $M_t := E[F|\mathcal{F}_t]$ has the representation

$$M_t = g(W_t, t), \quad 0 \leq t < T,$$

for a C^2 function $g : \mathbb{R} \times [0, T) \to \mathbb{R}$. Compute g in terms of the distribution function Φ of the standard normal distribution.

(b) Let (t_n) be a sequence of times such that $t_n \nearrow T$. Use Itô's formula to find predictable processes H^n such that

$$M^{t_n} - M_0 = H^n \bullet W, \qquad \text{for each } n \in \mathbb{N}.$$

Hint: Since M is a martingale, you do not need to calculate all the terms in Itô's formula.

(c) Find H such that $(\star)$ holds.

Solution 10.3

(a) We use the fact that $W_T - W_t \sim \mathcal{N}(0, T - t)$ is independent of $\mathcal{F}_t$. Therefore,

$$\begin{aligned} M_t &= P[a \leq W_T \leq b \mid \mathcal{F}_t] \\ &= P\left[\frac{a - W_t}{\sqrt{T-t}} \leq \frac{W_T - W_t}{\sqrt{T-t}} \leq \frac{b - W_t}{\sqrt{T-t}} \,\middle|\, \mathcal{F}_t\right] \\ &= \Phi\left(\frac{b - W_t}{\sqrt{T-t}}\right) - \Phi\left(\frac{a - W_t}{\sqrt{T-t}}\right) \\ &= g(W_t, t), \end{aligned}$$

where

$$g(x, t) = \Phi\left(\frac{b - x}{\sqrt{T-t}}\right) - \Phi\left(\frac{a - x}{\sqrt{T-t}}\right).$$

(b) By Itô's formula, and since M is a martingale, we have that

$$dM_t = \partial_x g(W_t, t) dW_t = \frac{1}{\sqrt{2\pi(T-t)}}\left(\exp\left(-\frac{(a - W_t)^2}{2(T-t)}\right) - \exp\left(-\frac{(b - W_t)^2}{2(T-t)}\right)\right) dW_t.$$

This holds on $[0, T)$, where g is C^2. In particular, we have that $M^{t_n} - M_0 = H^n \bullet W$, where

$$H^n_t = \mathbb{1}_{\{t\in[0,t_n]\}} \frac{1}{\sqrt{2\pi(T-t)}}\left(\exp\left(-\frac{(a - W_t)^2}{2(T-t)}\right) - \exp\left(-\frac{(b - W_t)^2}{2(T-t)}\right)\right).$$

(c) We claim that $(\star)$ holds, where H is defined by

$$H_t = \mathbb{1}_{t\in[0,T)} \frac{1}{\sqrt{2\pi(T-t)}}\left(\exp\left(-\frac{(a - W_t)^2}{2(T-t)}\right) - \exp\left(-\frac{(b - W_t)^2}{2(T-t)}\right)\right).$$

Since $P[W_T \in \mathbb{R} \setminus \{a, b\}] = 1$ and W is continuous a.s., it is easy to see that $\lim_{t \nearrow T} H_t = 0$ a.s. Thus, H is continuous, hence locally bounded, so that $H \bullet W$ is well defined and a continuous process. Likewise, we see that $M_t = g(W_t, t) \to \mathbb{1}_{\{a \leq W_T \leq b\}}$ a.s. as $t \nearrow T$. Since it holds that

$$M_{t_n} = (H^n \bullet W)_{t_n} = (H \bullet W)_{t_n}, \qquad \text{for each } n \in \mathbb{N},$$

we can take the limit $n \nearrow \infty$ to obtain that

$$M_T = H \bullet W_T.$$

Noting that $M_0 = E[F]$, $M_T = F$ and

$$H \bullet W_T = \int_0^T H_s dW_s = \int_0^\infty H_s dW_s$$

as $H_t = 0$ for $t \geq T$, we conclude that $(\star)$ holds.

Remark: We can show that $W_T - W_t$ and $\mathcal{F}_t$ are independent as follows. Let Z be a $\mathcal{F}_t$-measurable random variable. Since $\mathbb{F}$ is the augmentation of the raw filtration $\mathbb{F}^0$ generated by W, there exists some $\mathcal{F}_t^0$-measurable random variable $\tilde{Z}$ such that $Z = \tilde{Z}$ a.s. Therefore, for any bounded measurable function $f : \mathbb{R} \to \mathbb{R}$, we have that

$$E[Zf(W_T - W_t)] = E[\tilde{Z} f(W_T - W_t)] = E[\tilde{Z}] E[f(W_T - W_t)] = E[Z] E[f(W_T - W_t)],$$

which shows the claim.