Brownian Motion and Stochastic Calculus

Exercise sheet 11

Exercise 11.1 Let $\theta \in \mathbb{R}, \sigma > 0$ and $W = (W_t)_{t \geq 0}$ be a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

(a) Find a strong solution to the *Langevin equation*

$$dX_t = -\theta X_t \, dt + \sigma \, dW_t, \quad X_0 = x \in \mathbb{R}.$$

Hint: Consider $U_t = e^{\theta t} X_t$.

Remark: For $\theta > 0$, this SDE describes exponential convergence to zero "with noise".

(b) Show that there exists a Brownian motion B such that $Y := X^2$ satisfies the SDE

$$dY_t = (-2\theta Y_t + \sigma^2) \, dt + 2\sigma \sqrt{Y_t} \, dB_t. \tag{$\star$}$$

In other words, show that $(\Omega, \mathcal{F}, \mathbb{F}, P, B, Y)$ is a weak solution of the SDE $(\star)$.

Solution 11.1

(a) Itô's formula applied to the process $U = (U_t)_{t \geq 0}$ given by $U_t = e^{\theta t} X_t$ yields

$$dU_t = \theta e^{\theta t} X_t \, dt + e^{\theta t}(-\theta X_t \, dt + \sigma \, dW_t) = \sigma e^{\theta t} \, dW_t.$$

Thus, if X is a solution to the Langevin equation, then $U_t = U_0 + \int_0^t \sigma e^{\theta s} \, dW_s$. Conversely, if

$$X_t = e^{-\theta t} U_t = e^{-\theta t} \left(U_0 + \sigma \int_0^t e^{\theta s} \, dW_s \right) = e^{-\theta t} \left(x + \sigma \int_0^t e^{\theta s} \, dW_s \right),$$

then, by Itô's formula,

$$dX_t = -\theta X_t dt + \sigma e^{-\theta t} e^{\theta t} dW_t = -\theta X_t \, dt + \sigma \, dW_t,$$

so that X solves the Langevin equation with $X_0 = x$.

(b) By Itô's formula,

$$\begin{aligned} dY_t &= 2X_t(-\theta X_t \, dt + \sigma \, dW_t) + \sigma^2 \, dt \\ &= (-2\theta X_t^2 + \sigma^2) \, dt + 2\sigma X_t \, dW_t \\ &= (-2\theta Y_t + \sigma^2) \, dt + 2\sigma \sqrt{Y_t} \, \mathrm{sign}(X_t) \, dW_t. \end{aligned}$$

The result then follows from the fact that $B_t := \int_0^t \mathrm{sign}(X_s) \, dW_s$ is a Brownian motion, by Lévy's characterisation theorem.

Exercise 11.2 Let $(W_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider the SDE

$$dX_t = \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)\,dt \;+\; \sqrt{1+X_t^2}\,dW_t, \quad X_0 = x \in \mathbb{R}. \tag{1}$$

(a) Show that for any $x \in \mathbb{R}$, the SDE defined in (1) has a unique strong solution.

(b) Show that $(X_t)_{t\geq 0}$ defined by $X_t = \sinh\big(\sinh^{-1}(x) + t + W_t\big)$ is the unique solution of (1).
Hint: Consider the process $(Y_t)_{t\geq 0}$ defined by $Y_t := \sinh^{-1}(X_t)$.

Solution 11.2

(a) We see that the SDE is of the form

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t, \quad X_0 = x \in \mathbb{R},$$

where

$$a(x) = \sqrt{1+x^2} + \frac{x}{2} \quad \text{and} \quad b(x) = \sqrt{1+x^2}.$$

We observe that

$$\sup_{x\in\mathbb{R}} |b'(x)| = \sup_{x\in\mathbb{R}} \Big|\frac{x}{\sqrt{1+x^2}}\Big| \leq 1$$

as well as

$$\sup_{x\in\mathbb{R}} |a'(x)| = \sup_{x\in\mathbb{R}} \Big|\frac{x}{\sqrt{1+x^2}} + \frac{1}{2}\Big| \leq \frac{3}{2}.$$

Thus, by the mean value theorem, we obtain that $a(\cdot)$ and $b(\cdot)$ satisfy the Lipschitz condition

$$|a(y) - a(z)| + |b(y) - b(z)| \leq \frac{5}{2}|y - z|, \quad y, z \in \mathbb{R}.$$

Moreover, for any $x \in \mathbb{R}$,

$$\begin{aligned} |a(x)| &= \Big|\sqrt{1+x^2} + \frac{x}{2}\Big| \leq \Big|1 + |x| + \frac{x}{2}\Big| \leq \frac{3}{2}\big(1+|x|\big), \\ |b(x)| &= \Big|\sqrt{1+x^2}\Big| \leq 1 + |x|, \end{aligned}$$

so that $a(\cdot)$ and $b(\cdot)$ are also of linear growth. Thus, for any $x \in \mathbb{R}$, there exists a unique strong solution to (1) by Theorem 4.7.4 in the lecture notes.

(b) For $f := \sinh^{-1} \in C^2$ (the inverse function of the hyperbolic sine), we have the derivatives

$$f'(x) = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad f''(x) = -\frac{x}{(1+x^2)^{3/2}}.$$

Thus, if X solves (1), we can apply Itô's formula to $Y_t := f(X_t)$ to obtain that

$$dY_t = df(X_t) = dt + dW_t, \quad Y_0 = \sinh^{-1}(x),$$

which implies that $Y_t = \sinh^{-1}(x) + t + W_t$. Conversely, letting

$$X_t = \sinh\big(\sinh^{-1}(x) + t + W_t\big), \quad t \geq 0,$$

we have by Itô's formula that

$$\begin{aligned} dX_t &= \cosh\big(\sinh^{-1}(x) + t + W_t\big)(dt + dW_t) + \frac{1}{2}X_t dt \\ &= \Big(\sqrt{1+X_t^2} + \frac{1}{2}X_t\Big)\,dt \;+\; \sqrt{1+X_t^2}\,dW_t, \end{aligned}$$

so X solves (1). Alternatively, we can omit the converse computation and simply note from (a) that (1) has a unique solution.

Exercise 11.3

(a) Let $x_0 \in \mathbb{R}$, W be a Brownian motion and $(\mu_t)_{t\geq 0}$ a bounded predictable process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Show that there exists a unique adapted solution $(X_t)_{t\geq 0}$ to the equation

$$X_t = x_0 + \int_0^t \mu_s ds + \int_0^t X_s dW_s,$$

which is given by

$$X_t = \mathcal{E}(W)_t \left(x_0 + \int_0^t \left(\mathcal{E}(W)_s\right)^{-1} \mu_s ds \right).$$

In particular, if $x_0 \geq 0$ and $\mu_t \geq 0$ for all $t \geq 0$, then $X_t \geq 0$ for all $t \geq 0$.

(b) Let $x_1, x_2 \in \mathbb{R}$ and $a_1, a_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous functions that are Lipschitz and have linear growth, as in Theorem (4.7.4) of the lecture notes. Suppose that $x_1 \geq x_2$ and $a_1(t,x) \geq a_2(t,x)$ for all $t \geq 0, x \in \mathbb{R}$. Show that there exist unique solutions X^1 and X^2 to the SDEs

$$X_t^1 = x_1 + \int_0^t a_1(s, X_s^1) ds + \int_0^t X_s^1 dW_s,$$
$$X_t^2 = x_2 + \int_0^t a_2(s, X_s^2) ds + \int_0^t X_s^2 dW_s,$$

and that $X_t^1 \geq X_t^2$ for all $t \geq 0$ almost surely,

Hint: Argue that $a_2(s, X_s^1) - a_2(s, X_s^2) = \pi_s(X_s^1 - X_s^2)$, where π is a predictable process bounded by K (the Lipschitz constant). A Girsanov transformation may also be useful.

Solution 11.3

(a) To prove that the solution is unique, suppose that X and $\tilde{X}$ are two solutions. Then,

$$X_t - \tilde{X}_t = \int_0^t (X_s - \tilde{X}_s) dW_s.$$

Thus, $X - \tilde{X}$ satisfies a linear SDE, which has the unique solution

$$X_t - \tilde{X}_t = \mathcal{E}(W)_t (X_0 - \tilde{X}_0) = 0.$$

Therefore, $X = \tilde{X}$.

To show the given X is a solution, first note that it is adapted. We apply Itô's formula to the function $f(z_1, z_2) = z_1 (x_0 + z_2)$ with the processes $Z_t^1 = \mathcal{E}(W)_t$ and $Z_t^2 = \int_0^t \left(\mathcal{E}(W)_s\right)^{-1} \mu_s ds$. This gives

$$dX_t = \mathcal{E}(W)_t \big(\mathcal{E}(W)_t\big)^{-1} \mu_t dt + \left(x_0 + \int_0^t \left(\mathcal{E}(W)_s\right)^{-1} \mu_s ds \right) \mathcal{E}(W)_t dW_t = \mu_t dt + X_t dW_t,$$

as we wanted.

Since $\mathcal{E}(W)_t \geq 0$ for all $t \geq 0$, it is clear from the formula that $X_t \geq 0$ for all $t \geq 0$.

(b) Since a_1 and a_2 are Lipschitz and have linear growth, and likewise for $b(t,x) = x$, it follows that the SDEs have unique strong solutions. Denote $U_t = X_t^1 - X_t^2$ and $u = x_1 - x_2$. Taking the difference, we have that

$$U_t = u + \int_0^t \left(a_1(s, X_s^1) - a_2(s, X_s^2) \right) ds + \int_0^t U_s dW_s.$$

Note that we can write

$$\int_0^t \left(a_1(s, X_s^1) - a_2(s, X_s^2)\right) ds = \int_0^t \mu_s ds + \int_0^t \left(a_2(s, X_s^1) - a_2(s, X_s^2)\right) ds,$$

where $\mu_t := a_1(t, X_t^1) - a_2(t, X_t^1) \geq 0$ for all $t \geq 0$ a.s., by assumption. Since a_2 is Lipschitz, we can find a measurable function $\bar{\pi} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ that is bounded by K and such that

$$a_2(t, y_1) - a_2(t, y_2) = (y_1 - y_2)\bar{\pi}(t, y_1, y_2).$$

For example, where $y_1 \neq y_2$, we can simply set $\bar{\pi}(t, y_1, y_2) = \frac{a_2(t,y_1)-a_2(t,y_2)}{y_1-y_2}$. Letting $\pi_t = \bar{\pi}(t, X_t^1, X_t^2)$, we have that

$$U_t = u + \int_0^t \mu_s ds + \int_0^t \pi_s U_s ds + \int_0^t U_s dW_s.$$

Let $T > 0$. Since π is bounded, we have by Novikov's criterion that the density

$$\frac{dQ}{dP} = \mathcal{E}(-\pi \bullet W)_T$$

defines an equivalent measure Q. By Girsanov's theorem,

$$W_t^* = W_t + \int_0^t \mathbb{1}_{\{s \in [0,T]\}} d\langle \pi \bullet W, W\rangle_s = W_t + \int_0^{t \wedge T} \pi_s ds$$

is a Brownian motion under Q. Therefore, it holds under Q that

$$U_t = u + \int_0^t \mu_s ds + \int_0^t U_s dW_s^*, \quad 0 \leq t \leq T.$$

Since $u \geq 0$ and $\mu \geq 0$, it follows by (a) that $U = X^1 - X^2 \geq 0$ as well, as we wanted.

Exercise 11.4

(a) Let W be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and $\bar{f} : \mathbb{R} \to \mathbb{R}$ be a measurable bounded odd function. That is, $\bar{f}(-x) = -\bar{f}(x)$ or, equivalently,

$$\bar{f}(x) = f(x)\mathbb{1}_{\{x>0\}} - f(-x)\mathbb{1}_{\{x<0\}},$$

for some bounded measurable function $f : (0, \infty) \to \mathbb{R}$. Show that the process

$$Y = \bar{f}(W) \bullet W$$

is adapted with respect to the P-augmented filtration $\mathbb{F}^{|W|}$ generated by $|W|$ and all nullsets.

Hint: Start by considering $f(x) = \sin(\lambda x)$ for $\lambda > 0$ and applying Itô's theorem. Conclude by approximation.

(b) Let X be a Brownian motion and define $B = \operatorname{sign}(X) \bullet X$, so that B is a Brownian motion and the process $X = \operatorname{sign}(X) \bullet B$ solves the Tanaka equation as in Example (4.7.10). Show that X is not adapted to the filtration generated by B.

Solution 11.4

(a) Suppose that $f(x) = \sin(\lambda x)$. By Itô's formula applied to $\cos(\lambda x)$, we have that

$$-\frac{\cos(\lambda W_t)}{\lambda} = -\frac{1}{\lambda} + \int_0^t \sin(\lambda W_t) dW_t + \frac{1}{2}\int_0^t \lambda \cos(\lambda W_t) dt.$$

Rearranging gives

$$Y_t = \frac{1 - \cos(\lambda |W_t|)}{\lambda} - \frac{1}{2}\int_0^t \lambda \cos(\lambda |W_t|) dt,$$

using the fact that cosine is an even function. Therefore, $f(W) \bullet W_t$ is $\mathcal{F}_t^{|W|}$-measurable.

Since $f(W) \bullet W$ is linear in f, this property extends to all linear combinations

$$f_k(x) = \sum_{k=1}^{K} a_k \sin(\lambda_k x),$$

for $a_1, \ldots, a_k \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_k > 0$.

Let $f : (0, \infty) \to \mathbb{R}$ be measurable and bounded by $C > 0$. Note that W is predictable as it is adapted and continuous, so that $f(W)$ is also predictable because f is measurable. By properties of Fourier series, we can find a sequence (f_k) as above such that $\|f_k\|_\infty \leq C$ for each k and $f_k(x) \to f(x)$ for Lebesgue-almost all $x > 0$ (one way to show this is to approximate f in $L^2((0, n))$ for each $n \in \mathbb{N}$ and take a diagonal sequence).

Let $A = \{x > 0 : f_k(x) \to f(x)\}$, so that $A^c = \mathbb{R}_+ \setminus A$ has Lebesgue measure 0. Since the sequence (f_k) is uniformly bounded and converges pointwise to f on A, it follows from Theorem 4.2.19 that for each $t \geq 0$,

$$\int_0^t \big(f_k(W_s) - f(W_s)\big)\mathbb{1}_{\{W_s \in A\}} dW_s \to 0$$

in probability. Also, note that by Fubini,

$$E\left[\int_0^\infty \mathbb{1}_{\{W_s \in A^c\}} d\langle W \rangle_s\right] = \int_0^\infty P[W_s \in A^c] ds = 0,$$

since $P[W_s \in A^c] = 0$ as W_s has a continuous density on $\mathbb{R}$ with respect to Lebesgue measure for each $s > 0$. Noting that the f_k and f are bounded, it follows by construction of the stochastic integral that

$$\int_0^t \big(f_k(W_s) - f(W_s)\big)\mathbb{1}_{\{W_s \in A^c\}} dW_s = 0$$

for all $t \geq 0$ and $k \in \mathbb{N}$ a.s. Adding the two cases, we conclude that for each $t \geq 0$

$$\int_0^t f_k(W_s)\mathbb{1}_{\{W_s>0\}} dW_s \to \int_0^t f(W_s)\mathbb{1}_{\{W_s>0\}} dW_s$$

in probability as $k \to \infty$. We can similarly show that

$$\int_0^t -f_k(-W_s)\mathbb{1}_{\{W_s<0\}} dW_s \to \int_0^t -f(-W_s)\mathbb{1}_{\{W_s<0\}} dW_s$$

in probability, therefore

$$Y_t^k := \int_0^t \bar{f}_k(W_s) dW_s \to \int_0^t \bar{f}(W_s) dW_s = Y_t$$

in probability. By taking a subsequence, we can take the convergence to be P-almost sure. By construction, each Y_t^k is $\mathcal{F}_t^{|W|}$-measurable, so the same is true of the limit Y_t. This shows the result.

(b) By (a) applied to $(W, Y) := (X, B)$, and since sign is an odd function, we have that B is adapted with respect to the filtration $\mathbb{F}^{|X|} \subseteq \mathbb{F}^X$. In other words, $\mathcal{F}_t^B \subseteq F_t^{|X|} \subseteq F_t^X$ for each $t \geq 0$. It remains to show that X is not adapted to $\mathcal{F}^{|X|}$. One argument is as follows. We claim that for all bounded $\mathcal{F}_t^{|X|}$-measurable random variables Z, it holds that

$$E[X_t Z] = E[-X_t Z] = 0. \qquad (\star\star)$$

Indeed, if Z has the form

$$Z = \prod_{j=1}^J f_j(|X_{s_j}|)$$

for $0 \leq s_1 < \cdots < s_j \leq t$ and bounded measurable functions f_j, then

$$E[X_t Z] = E\left[X_t \prod_{j=1}^J f_j(|X_{s_j}|)\right] = E\left[-X_t \prod_{j=1}^J f_j(|-X_{s_j}|)\right] = E[-X_t Z],$$

using the fact that $-X$ is a Brownian motion and therefore has the same law as X. The claim then follows by an easy application of the monotone class theorem.

As a consequence of $(\star\star)$, it holds that

$$E\Big[X_t \,\Big|\, \mathcal{F}_t^{|X|}\Big] = E\Big[-X_t \,\Big|\, \mathcal{F}_t^{|X|}\Big] = 0 \quad \text{a.s.}$$

But this cannot be the case if X_t is $\mathcal{F}_t^{|X|}$ measurable, since $X_t \neq 0$ a.s. Therefore, X cannot be adapted with respect to $(\mathcal{F}_t^{|X|})$ nor $(\mathcal{F}_t^B)$.