Brownian Motion and Stochastic Calculus

Exercise sheet 12

Exercise 12.1 Consider the SDE

$$\begin{aligned} dX_t^x &= a(X_t^x)\,dt + b(X_t^x)\,dW_t, \\ X_0^x &= x, \end{aligned} \tag{1}$$

where W is an $\mathbb{R}^m$-valued Brownian motion and the functions $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable and locally bounded (i.e., bounded on compact sets). Let $U \subseteq \mathbb{R}^d$ be a bounded open set such that the stopping time $T_U^x := \inf\{s \geq 0 : X_s^x \notin U\}$ is P-integrable for all $x \in U$. Consider the boundary problem

$$\begin{aligned} Lu(x) + c(x)u(x) &= -f(x) \qquad \text{for } x \in U, \\ u(x) &= g(x) \qquad \text{for } x \in \partial U, \end{aligned} \tag{2}$$

where $c, f \in C_b(U)$ and $g \in C_b(\partial U)$ are given functions such that $c \leq 0$ on U, and the linear operator L is defined by

$$Lf(x) := \sum_{i=1}^d a_i(x)\,\frac{\partial f}{\partial x^i}(x) + \frac{1}{2}\sum_{i,j=1}^d \big(b\,b^\top\big)_{ij}(x)\,\frac{\partial^2 f}{\partial x^i\,\partial x^j}(x).$$

Suppose that $(X_t^x)_{t\geq 0}$ solves the SDE (1) for some $x \in U$ and $u \in C^2(U) \cap C(\bar{U})$ is a solution to the boundary problem (2). Show that

$$u(x) = E\bigg[g(X^x_{T_U^x})\,\exp\bigg(\int_0^{T_U^x} c(X_s^x)\,ds\bigg)\bigg] + E\bigg[\int_0^{T_U^x} f(X_s^x)\,\exp\bigg(\int_0^s c(X_r^x)\,dr\bigg)\,ds\bigg].$$

Solution 12.1 For each $n \in \mathbb{N}$ such that $\operatorname{dist}(x, U^c) > 1/n$, define the stopping time

$$T_n := \inf\left\{s \geq 0 : \operatorname{dist}(X_s^x, U^c) \leq \frac{1}{n}\right\}.$$

There exist functions $u_n \in C^2(\mathbb{R}^d;\mathbb{R})$ such that $u = u_n$ on the set $\{z \in U : \operatorname{dist}(z, U^c) \geq \frac{1}{n}\}$. (see the remark at the end). Let

$$Y_t^n := u_n(X_t^x)\,\exp\bigg(\int_0^t c(X_s^x)\,ds\bigg).$$

By Itô's formula, we have that

$$Y_t^n = u_n(x) + \int_0^t \exp\bigg(\int_0^s c(X_r^x)dr\bigg)\Big(\big(Lu_n(X_s^x) + c(X_s^x)u_n(X_s^x)\big)\,ds + \nabla u_n(X_s^x)b(X_s^x)dW_s\Big).$$

As b and c are bounded on $U \subseteq \mathbb{R}$ and $u_n \in C_c^2$, we can easily check that the process

$$M_t^n = \int_0^{t\wedge T_n} \exp\bigg(\int_0^s c(X_r^x)dr\bigg)\,\nabla u_n(X_s^x)b(X_s^x)dW_s$$

is in $\mathcal{H}_0^{2,c}$ (since $(X^x)^{T_n}$ does not leave U), so that M^n is a true martingale. Taking expectations, we obtain

$$E\left[Y^n_{t\wedge T_n}\right] - u_n(x) = E\bigg[\int_0^{t\wedge T_n} \big(Lu_n(X_s^x) + c(X_s^x)u_n(X_s^x)\big)\exp\bigg(\int_0^s c(X_r^x)\,dr\bigg)\,ds\bigg].$$

By the definition of T_n^x, we have $u_n(X^x_{t\wedge T_n}) = u(X^x_{t\wedge T_n})$ for $t \geq 0$, as $\mathrm{dist}(X^x_{t\wedge T_n}, U^c) \geq \frac{1}{n}$. Moreover, $u_n(x) = u(x)$ as $\mathrm{dist}(x, U^c) > 1/n$. Since u solves (2), we get

$$u(x) = E\bigg[u(X^x_{t\wedge T_n}) \exp\bigg(\int_0^{t\wedge T_n} c(X^x_s)\, ds\bigg)\bigg] + E\bigg[\int_0^{t\wedge T_n} f(X^x_s) \exp\bigg(\int_0^s c(X^x_r)\, dr\bigg)\, ds\bigg]. \quad (3)$$

By continuity of the process $(\mathrm{dist}(X^x_t, U^c))$, we have that $T_n \nearrow T^x_U < \infty$, which is integrable by assumption. Since $c \leq 0$, for any $n \in \mathbb{N}$ and $t \geq 0$, we have that

$$\bigg|u(X^x_{t\wedge T_n}) \exp\bigg(\int_0^{t\wedge T_n} c(X^x_s)\, ds\bigg)\bigg| \leq \sup_{y\in \bar{U}} |u(y)| < \infty,$$
$$\bigg|\int_0^{t\wedge T_n} f(X^x_s) \exp\bigg(\int_0^s c(X^x_r) dr\bigg)\bigg| \leq T^x_U \sup_{y\in \bar{U}} |f(y)|.$$

Note that $X^x_{T^x_U} \in \partial U$ by the definition of T^x_U, so that $u(X^x_{T^x_U}) = g(X^x_{T^x_U})$ by (2). By the dominated convergence theorem, we can let $t \to \infty$ and $n \to \infty$ in (3) to conclude

$$u(x) = E\bigg[g(X^x_{T^x_U}) \exp\bigg(\int_0^{T^x_U} c(X^x_s)\, ds\bigg)\bigg] + E\bigg[\int_0^{T^x_U} f(X^x_s) \exp\bigg(\int_0^s c(X^x_r)\, dr\bigg)\, ds\bigg].$$

Remark: In order to find the smooth extensions u_n, we can use the following lemma.

Lemma 1. *Let $K \subseteq \mathbb{R}^n$ be compact and $C \subseteq \mathbb{R}^n$ be closed such that $C \cap K = \emptyset$. Then, there exists a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\psi(x) = 1$ for all $x \in K$ and $\psi(x) = 0$ for all $x \in C$.*

Proof. The set K is covered by the union of open balls

$$\bigcup \{B_\varepsilon(x) : x \in K \text{ and } \varepsilon \in (0, \mathrm{dist}(x, C)\},$$

so that each ball satisfies $B_\varepsilon(x) \cap C = \emptyset$. Since K is compact, there exists a finite subcover by open balls $B_j := B_{\varepsilon_j}(x_j)$, where $j = 1, \dots, J$. We can find "bump functions" on each B_j, i.e., smooth functions $\rho_j : \mathbb{R}^n \to \mathbb{R}$ such that $\rho_j(x) > 0$ for all $x \in B_j$ and $\rho_j(x) = 0$ for all $x \in \mathbb{R}^n \setminus B_j$.

Let $\tilde{\psi} := \sum_{j=1}^J \rho_j$. By construction, we have that $\tilde{\psi}$ is nonnegative, with $\tilde{\psi}(x) > 0$ for all $x \in K$ while $\tilde{\psi}(x) = 0$ for all $x \in C$. Since K is compact, we have that $\inf_{x\in K} \psi(x) = c_0 > 0$. We can construct a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(x) = 0$ for all $x \leq 0$ and $\phi(x) = 1$ for all $x \geq c_0$. Setting $\psi(x) = \phi(\tilde{\psi}(x))$, it is easy to check that ψ satisfies the required properties. □

In the setting of the exercise, let

$$K := \{x \in U : \mathrm{dist}(x, U^c) \geq 1/n\},$$
$$C := \{x \in U : \mathrm{dist}(x, U^c) \leq 1/(n+1)\} \supseteq \overline{U^c},$$

with corresponding ψ_n given by the lemma. We can find an extension $u_n : \mathbb{R}^n \to \mathbb{R}$ as

$$u_n(x) = u(x)\psi_n(x),$$

where $u_n(x) = 0$ for $x \in U^c \subseteq C$. It is easy to check that u_n satisfies the required properties.

Exercise 12.2

(a) Show that the spaces $\mathcal{R}^2$ and $\mathcal{A}^2$ defined in page 174 are Banach spaces.

(b) Let β, γ be bounded predictable processes and define

$$Y = \mathcal{E}\left(-\int \beta_s ds - \int \gamma_s dW_s\right).$$

Show that, for any $T > 0$, the random variable

$$Y_T^* := \sup_{0 \le s \le T} |Y_s|$$

is in L^p for every $p < \infty$ (see page 180).

Solution 12.2

(a) It is clear that $\mathcal{R}^2$ is a normed space (up to equivalence classes) with norms $\|\cdot\|_{\mathcal{R}^2}$. This follows from the corresponding norm properties of the sup- and L^2-norms. For example, to show subadditivity, we have for $X, Y \in \mathcal{R}^2$ that

$$\|X+Y\|_{\mathcal{R}^2} = \|(X+Y)_T^*\|_{L^2(P)} \le \|X_T^*+Y_T^*\|_{L^2(P)} \le \|X_T^*\|_{L^2(P)}+\|Y_T^*\|_{L^2(P)} = \|X\|_{\mathcal{R}^2}+\|Y\|_{\mathcal{R}^2}.$$

To prove that $\mathcal{R}^2$ is complete, let (X^n) be a Cauchy sequence in $\mathcal{R}^2$. Let (X^{n_m}) be an arbitrary subsequence. Note that, for $k \ge 0$, we have by the Markov inequality that

$$\|(X-Y)_T^*\|_{L^2(P)} \le 2^{-2k} \Rightarrow P[(X-Y)_T^* > 2^{-k}] \le 2^{-2k}.$$

Therefore, since (X^{n_m}) is Cauchy in $\mathcal{R}^2$, we can find a further subsequence $(X^{n_{m_k}})$ such that

$$P\left[(X^{n_{m_{k+1}}} - X^{n_{m_k}})_T^* > 2^{-k}\right] \le 2^{-2k}$$

for each $k \in \mathbb{N}$. By Borel–Cantelli, we have that $P[A] = 1$, where

$$A = \bigcup_{K\in\mathbb{N}} \bigcap_{k\ge K} \left\{(X^{n_{m_{k+1}}} - X^{n_{m_k}})_T^* \le 2^{-k}\right\}.$$

This implies that, for P-almost all $\omega \in \Omega$, $(X_\cdot^{n_{m_k}}(\omega))$ is a Cauchy sequence of RCLL functions with respect to the uniform norm. Since the space of RCLL functions is a Banach space, for P-almost all ω there exists an RCLL function $X(\omega)$ such that $X^{n_{m_k}}(\omega) \to X(\omega)$ uniformly as $k \to \infty$. Define X arbitrarily on the remaining nullset. We claim that $X \in \mathcal{R}^2$ and $X^{n_{m_k}} \to X$ in $\mathcal{R}^2$.

For each $t \in [0, T]$ and almost all $\omega \in \Omega$, we have that $X_t(\omega) = \lim_{k\to\infty} X_t^{n_{m_k}}(\omega)$. Since each $X^{n_{m_k}}$ is adapted and the filtration satisfies the usual conditions (in particular, it is P-complete), we get that X is adapted. We also know that X is RCLL by construction. By uniform convergence, we have that $X_T^*(\omega) = \lim_{k\to\infty}(X^{n_{m_k}})_T^*(\omega)$ for almost all ω. Therefore, by Fatou's lemma and the Cauchy property,

$$E\left[(X_T^*)^2\right] = E\left[\lim_{k\to\infty}((X^{n_{m_k}})_T^*)^2\right] \le \liminf_{k\to\infty} E\left[((X^{n_{m_k}})_T^*)^2\right] < \infty.$$

This shows that $X \in \mathcal{R}^2$.

To prove that $X^{n_{m_k}} \to X$ in $\mathcal{R}^2$, it suffices to show that $|X^{n_{m_k}} - X|_T^* \to 0$ in $L^2(P)$. Indeed, each $|X^{n_{m_k}} - X|_T^* \in L^2(P)$ since $X^{n_{m_k}}, X \in \mathcal{R}^2$. Moreover, the sequence $(|X^{n_{m_k}} - X|_T^*)$ is Cauchy in L^2. Indeed, for $k, k' \ge K$ we have by the reverse triangle inequality

$$\|((X^{n_{m_k}} - X)_T^* - (X^{n_{m_{k'}}} - X)_T^*)\|_{L^2} \le \|(X^{n_{m_k}} - X^{n_{m_{k'}}})_T^*\|_{L^2} = \|X^{n_{m_k}} - X^{n_{m_{k'}}}\|_{\mathcal{R}^2} \to 0$$

uniformly over k, k' as $K \to \infty$, since $(X^{n_{m_k}})$ is Cauchy in $\mathcal{R}^2$. By completeness of L^2, the sequence $(|X^{n_{m_k}} - X|_T^*)$ has a limit in L^2. Since it also converges to 0 almost surely, we have that $|X^{n_{m_k}} - X|_T^* \to 0$ in $L^2(P)$, so we conclude that $X^{n_{m_k}} \to X$ in $\mathcal{R}^2$, as we wanted. We showed that any subsequence (X^{n_m}) has a further subsequence $(X^{n_{m_k}})$ that converges in $\mathcal{R}^2$, therefore the original Cauchy sequence (X^n) converges in $\mathcal{R}^2$.

To show that $\mathcal{A}^2$ is a Banach space, we just need to identify

$$\mathcal{A}^2 = L^2(\Omega \times [0,T], \mathcal{P}, P \otimes dt),$$

where $\mathcal{P}$ is the predictable σ-algebra. Since this is an L^2 space, we know that it is complete.

(b) Let β and γ be bounded by constants $B, C > 0$ respectively. For any $p \geq 1$ we have that

$$E\left[\exp\left(\frac{\langle -p\gamma \bullet W\rangle_T}{2}\right)\right] = E\left[\exp\left(\frac{p^2}{2}\int_0^T |\gamma_s|^2 ds\right)\right] \leq \exp\left(\frac{TC^2p^2}{2}\right) < \infty,$$

so by Novikov's criterion, $\mathcal{E}(-p\gamma \bullet W)$ is a true martingale. Therefore, we have that

$$\begin{aligned} E\left[\mathcal{E}(-\gamma \bullet W)_T^p\right] &= E\left[\exp\left(-\int_0^T p\gamma_s dW_s - \int_0^T \frac{p\gamma_s^2}{2} ds\right)\right] \\ &\leq \exp\left(\frac{T(p^2-p)C^2}{2}\right) E[\mathcal{E}(-p\gamma \bullet W)_T] \\ &= \exp\left(\frac{T(p^2-p)C^2}{2}\right). \end{aligned}$$

By Doob's L^p-inequality, we obtain that

$$E\left[\sup_{0\leq t\leq T} \mathcal{E}(-\gamma \bullet W)_t^p\right] \leq \left(\frac{p}{p-1}\right)^p \exp\left(\frac{T(p^2-p)C^2}{2}\right).$$

Therefore, bounding the finite variation term we conclude

$$\begin{aligned} E[(Y_T^*)^p] &= E\left[\sup_{t\in[0,T]} \mathcal{E}\left(-\int \beta_s ds - \int \gamma_s dW_s\right)_t^p\right] \\ &\leq \left(\frac{p}{p-1}\right)^p \exp\left(TpB + \frac{T(p^2-p)C^2}{2}\right) < \infty. \end{aligned}$$

Exercise 12.3 Consider a probability space $(\Omega, \mathcal{F}, P)$ supporting a Brownian motion $W = (W_t)_{t\geq 0}$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the P-augmentation of the (raw) filtration generated by W. Let $T > 0$, $\alpha > 0$ and let F be a bounded, $\mathcal{F}_T$-measurable random variable.

(a) Show that the process $X = (X_t)_{0\leq t\leq T}$ given by

$$X_t = -\alpha \log E[\exp(-F/\alpha) \mid \mathcal{F}_t]$$

solves the BSDE

$$\begin{aligned} dX_t &= \frac{1}{2\alpha} Z_t^2 dt + Z_t dW_t, \\ X_T &= F. \end{aligned}$$

Hint: We have that $X_t = -\alpha \log Y_t$, where $Y_t := E[\exp(-F/\alpha)|\mathcal{F}_t]$. Apply Itô's representation theorem to Y_T and Itô's formula to X to derive a solution pair $(X, Z) \in \mathcal{R}^2 \times L^2(W)$ for the BSDE.

Remark: Note that the generator of this BSDE is not Lipschitz, but quadratic in Z.

(b) Let $b \in \mathbb{R}$. Show that the process $X = (X_t)_{0\leq t\leq T}$ given by

$$X_t = -\alpha \left(\frac{1}{2} b^2 (t - T) - b\, W_t + \log E[\exp(bW_T - F/\alpha) \mid \mathcal{F}_t] \right)$$

solves the BSDE

$$\begin{aligned} dX_t &= \left(\frac{1}{2\alpha}\, Z_t^2 - b\, Z_t \right) dt + Z_t dW_t, \\ X_T &= F. \end{aligned}$$

Solution 12.3

(a) Itô's representation theorem applied to the bounded random variable $\exp(-F/\alpha)$ gives a unique representation

$$\exp(-F/\alpha) = E[\exp(-F/\alpha)] + \int_0^T H_s dW_s$$

for some $H \in L^2_{\text{loc}}(W)$ such that $H \bullet W$ is a true martingale. Since F is bounded, so is $\exp(-F/\alpha)$. Therefore, the continuous martingale $(Y_t)_{t\in[0,T]}$ defined by

$$Y_t = \int_0^t H_s dW_s + E[\exp(-F/\alpha)] = E[\exp(-F/\alpha) \mid \mathcal{F}_t]$$

is bounded as well. In particular, we have that $(H \bullet W)^T \in \mathcal{H}_0^{2,c}$, so $H \in L^2(W^T)$. Next, applying Itô's formula to $X_t = -\alpha \log Y_t$ and setting $Z_t := -\frac{\alpha H_t}{Y_t}$ yields

$$dX_t = -\frac{\alpha dY_t}{Y_t} + \frac{\alpha d\langle Y\rangle_t}{2Y_t^2} = -\frac{\alpha H_t}{Y_t}\, dW_t + \frac{\alpha H_t^2}{2Y_t^2} dt = Z_t\, dW_t + \frac{1}{2\alpha} Z_t^2\, dt,$$

and it only remains to show that $(X, Z) \in \mathcal{R}^2 \times L^2(W^T)$. Since F is bounded, we have that $c \leq Y \leq C$ for some constants $0 < c < C < \infty$. Hence, X is also bounded and thus $X \in \mathcal{R}^2$. Since Y is bounded away from 0, we have that $Z \in L^2(W^T)$ as $H \in L^2(W^T)$.

(b) Consider the measure $Q \approx P$ with density process

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = e^{bW_t - \frac{1}{2}b^2 t}, \quad 0 \le t \le T.$$

By Girsanov's theorem, we obtain that $W_t^Q = W_t - bt$ is a Q-Brownian motion. Moreover, on $[0,T]$, W and W^Q generate the same filtration. We can rewrite the BSDE as

$$\begin{aligned} dX_t &= \frac{1}{2\alpha} Z_t^2 dt + Z_t dW_t^Q, \\ X_T &= F. \end{aligned}$$

Under Q, the BSDE is as in (a). Thus, we deduce that

$$X_t = -\alpha \log E_Q[\exp(-F/\alpha) \mid \mathcal{F}_t]$$

is a solution. Using the definition of Q and Bayes' formula (see Proposition 4.4.4 in the script), we obtain that

$$\begin{aligned} X_t &= -\alpha \log E_Q[\exp(-F/\alpha) \mid \mathcal{F}_t] \\ &= -\alpha \log \left(e^{-bW_t + \frac{1}{2}b^2 t} \, E[e^{bW_T - \frac{1}{2}b^2 T} \, \exp(-F/\alpha) \mid \mathcal{F}_t] \right) \\ &= -\alpha \left(\frac{b^2(t-T)}{2} - b\, W_t + \log E[\exp(bW_T - F/\alpha) \mid \mathcal{F}_t] \right). \end{aligned}$$