ETH Zürich, Spring 2022
D-MATH
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# Brownian Motion and Stochastic Calculus

## Exercise sheet 13

**Exercise 13.1** Let $X$ be a Lévy process with values in $\mathbb{R}^d$ and $f_t(u) := E[e^{\mathrm{i}u^\top X_t}]$. Recall that $X$ is stochastically continuous, i.e., the map $t \mapsto X_t$ is continuous in probability, and that $f_{t+s}(u) = f_t(u) f_s(u)$ and $f_0(u) = 1$ for all $s, t \geq 0$ and $u \in \mathbb{R}^d$.

(a) Show that $f_s(u)^n = f_{ns}(u)$ and $f_t(u) = f_{t/n}(u)^n$ for all $n \in \mathbb{N}$ and $s, t \geq 0$.

(b) Show that $t \mapsto f_t(u)$ is right-continuous and $f_t(u) \neq 0$ for all $t \geq 0$ and $u \in \mathbb{R}^d$.

(c) Fix $u \in \mathbb{R}^d$ and let $\tilde{z} \in \mathbb{C}$ be such that $f_1(u) = \exp(\tilde{z})$. Show that there exists a unique $\hat{k} \in \mathbb{Z}$ such that

$$f_{2^{-n}}(u) = \exp\left(\frac{\tilde{z} + 2\hat{k}\pi\mathrm{i}}{2^n}\right)$$

for each $n \in \mathbb{N}$.

(d) In the setup of (e), let $z := \tilde{z} + 2\hat{k}\pi\mathrm{i}$ and define the function $g(t) = \exp(tz)$ (which can be seen as a definition of $t \mapsto f_1(u)^t$). Show that $f_t(u) = g(t)$ for all $t \geq 0$.

**Solution 13.1**

(a) It follows by induction on $n$ that $f_s(u)^n = f_{ns}(u)$ for any $s \geq 0$ and $n \in \mathbb{N}$. The second claim follows by setting $s = t/n$.

(b) Right-continuity of $t \mapsto f_t(u)$ follows immediately from right-continuity of $X$ and the dominated convergence theorem. Assume that $f_t(u) = 0$ for some $t > 0$ and $u \in \mathbb{R}^d$. Then it follows that $f_{t/n}(u)^n = f_t(u) = 0$, so that $f_{t/n}(u) = 0$ for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain a contradiction to the right-continuity at 0 because $f_0(u) = 1$.

(c) Since $f_1(u) \neq 0$, there exists such a $\tilde{z} \in \mathbb{C}$. For $n \in \mathbb{N}_0$, let

$$A_n = \left\{ k \in \mathbb{Z} : f_{2^{-m}}(u) = \exp\left(\frac{\tilde{z} + 2\hat{k}\pi\mathrm{i}}{2^m}\right) \text{ for each } m = 0, \dots, n \right\}.$$

Since $\exp(2\pi\mathrm{i}) = 1$, it is clear that $A_0 = \mathbb{Z}$. Since $f_{1/2}(u)^2 = f_1(z)$, it is easy to check that either $A_1 = 2\mathbb{Z}$ or $A_1 = 1 + 2\mathbb{Z}$. In other words, $A_1$ is an element of $\mathbb{Z}/2\mathbb{Z}$. Likewise, $A_2 \in \{m + 4\mathbb{Z} : m = 0, 1, 2, 3\} = \mathbb{Z}/4\mathbb{Z}$, and so on. Let $k_n$ be the element of $A_n$ with minimum norm (we take the positive one if there are two such elements). We claim that $(k_n)_{n\in\mathbb{N}}$ converges stationarily, i.e., there exists $N \in \mathbb{N}$ such that $k_n = k_N$ for all $n \geq N$. This implies that $\hat{k} = k_N = \lim_{n\to\infty} k_n$ exists and satisfies the requirement.

Since $A_0 \supseteq A_1 \supseteq \cdots$ and each $A_n \in \mathbb{Z}/2^n\mathbb{Z}$, we can see that $k_n \in \{k_{n-1}, k_{n-1} + 2^{n-1}\}$ if $k_n \leq 0$, or $k_n \in \{k_{n-1} - 2^{n-1}, k_{n-1}\}$ if $k_{n-1} > 0$. This implies that $|k_n| \leq 2^{n-1}$ and also that if $k_n \neq k_{n-1}$, then

$$|k_n| \geq 2^{n-1} - |k_{n-1}| \geq 2^{n-1} - 2^{n-2} = 2^{n-2}.$$

We can now show that $(k_n)$ converges stationarily. By (b), $t \mapsto f_t(u)$ is right-continuous at $t = 0$ with $f_0(u) = 1$, which implies that $\lim_{n\to\infty} f_{2^{-n}}(u) = 1$. Since $2^{-n}\tilde{z} \to 0$, it follows that

$$\lim_{n\to\infty} \exp\left(\frac{2k_n\pi\mathrm{i}}{2^n}\right) = \lim_{n\to\infty} f_{2^{-n}}(u) \exp(-2^{-n}\tilde{z}) = 1.$$

Since $|k_n| \leq 2^{n-1}$, we have that $\left|\frac{2k_n\pi}{2^n}\right| \leq \pi$. This is smaller than $2\pi$, which implies that the exponent must converge to 0, i.e., $2^{-n}k_n \to 0$. However, for any $n \in \mathbb{N}$ such that $k_n \neq k_{n-1}$, we have that $|2^{-n}k_n| \geq 2^{-n}2^{n-2} = 1/4$. Therefore, there exist only finitely many $n$ such that $k_n \neq k_{n-1}$, thus $(k_n)$ converges stationarily to a limit $\hat{k}$. Since $k_n = \hat{k}$ for all large enough $n$, we have that $\hat{k}$ satisfies the required property.

It is clear that $\hat{k}$ is unique, since for any other $\tilde{k}$ we have that $|2(\hat{k}-\tilde{k})\pi i/2^n| < 2\pi$ for large enough $n$. This implies that

$$\exp\left(\frac{\tilde{z} + 2\hat{k}\pi\mathrm{i}}{2^n}\right) \neq \exp\left(\frac{\tilde{z} + 2\tilde{k}\pi\mathrm{i}}{2^n}\right),$$

so that the equation for $f_{2^{-n}}(u)$ cannot hold for both $\hat{k}$ and $\tilde{k}$.

(d) By (c), we have that $g(2^{-n}) = f_{2^{-n}}(u)$ for each $n \in \mathbb{N}$, and we also have that $g(0) = f_0(u) = 1$. We get by (a) that

$$f_{m2^{-n}}(u) = f_{2^{-n}}(u)^m = \exp(2^{-n}z)^m = \exp(m2^{-n}z) = g(m2^{-n})$$

for all $m, n \in \mathbb{N}$. Note that the set $\{m2^{-n} : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}_+$. Since $g$ is continuous and $t \mapsto f_t(u)$ is right-continuous by (b), we can take right limits to show that $f_t(u) = g(t)$ for all $t \geq 0$, as we wanted.

**Exercise 13.2**

(a) Let $N$ be a one-dimensional Poisson process and $(Y_i)_{i\geq 1}$ a sequence of i.i.d. $\mathbb{R}^d$-valued random variables independent of $N$. We define the *compound Poisson process* by $X_t := \sum_{j=1}^{N_t} Y_j$. Show that $X$ is a Lévy process and calculate its Lévy triplet.

(b) Does there exist a Lévy process $X$ such that $X_1$ is uniformly distributed on $[0,1]$?

(c) Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be $\mathbb{R}^d$-valued processes such that the joint process $(X,Y)$ is Lévy with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)$. Show that if $E[e^{\mathrm{i}u^\top X_t} e^{\mathrm{i}v^\top Y_t}] = E[e^{\mathrm{i}u^\top X_t}]\, E[e^{\mathrm{i}v^\top Y_t}]$ for all $u, v \in \mathbb{R}^d$ and $t \geq 0$, then $X$ and $Y$ are independent.

**Solution 13.2**

(a) Define the discrete-time process $(\tilde{X}_n)_{n\in\mathbb{N}_0}$ by $\tilde{X}_n = \sum_{j=1}^n Y_j$, with natural filtration given by $\tilde{\mathcal{F}}_n = \sigma(Y_1,\dots,Y_n)$. It is clear that $\tilde{X}_0 = 0$ and $\tilde{X}$ has stationary and independent increments. We also know that the Poisson process $N$ is a Lévy process independent from $\tilde{X}$. In particular, $\mathcal{F}^N_\infty = \sigma(N_t : t \geq 0)$ and $\tilde{\mathcal{F}}_\infty = \sigma(Y_1, Y_2, \dots)$ are independent $\sigma$-algebras. We need to show that the process $(X_t)_{t\geq 0}$ defined by

$$X_t = \sum_{j=1}^{N_t} Y_j = \tilde{X}_{N_t}$$

is Lévy. For $0 \leq t_1 < \dots < t_m$ and bounded measurable functions $f_j$, define the function $g_j(n) := E[f_j(\tilde{X}_n)]$. Using the properties of $\tilde{X}$ and $N$, we have that

$$\begin{aligned}
E\left[\prod_{j=1}^m f_j(X_{t_j} - X_{t_{j-1}})\right] &= E\left[\prod_{j=1}^m f_j(\tilde{X}_{N_{t_j}} - \tilde{X}_{N_{t_{j-1}}})\right] \\
&= E\left[E\left[\prod_{j=1}^m f_j(\tilde{X}_{N_{t_j}} - \tilde{X}_{N_{t_{j-1}}}) \,\middle|\, \mathcal{F}^N_\infty\right]\right] \\
&= E\left[E\left[\prod_{j=1}^m f_j(\tilde{X}_{n_j} - \tilde{X}_{n_{j-1}})\right]\Bigg|_{n_j = N_{t_j}}\right] \\
&= E\left[\prod_{j=1}^m E\left[f_j(\tilde{X}_{n_j} - \tilde{X}_{n_{j-1}})\right]\big|_{n_j = N_{t_j}}\right] \\
&= E\left[\prod_{j=1}^m E\left[f_j(\tilde{X}_{n_j - n_{j-1}})\right]\big|_{n_j = N_{t_j}}\right] \\
&= E\left[\prod_{j=1}^m g_j(N_{t_j} - N_{t_{j-1}})\right] = \prod_{j=1}^m E\big[g_j(N_{t_j} - N_{t_{j-1}})\big] \\
&= \prod_{j=1}^m E\big[g_j(N_{t_j - t_{j-1}})\big] = \prod_{j=1}^m E\big[E[f_j(\tilde{X}_{n_j})]|_{n_j = N_{t_j - t_{j-1}}}\big] \\
&= \prod_{j=1}^m E\Big[E\big[f_j(\tilde{X}_{N_{t_j - t_{j-1}}}) \mid \mathcal{F}^N_\infty\big]\Big] = \prod_{j=1}^m E\big[f_j(X_{t_j - t_{j-1}})\big].
\end{aligned}$$

As in the solution to Exercise **6.2**, this shows that $X$ is Lévy (also noting that $X_0 = 0$).

Next, we calculate the Lévy triplet. For $u \in \mathbb{R}^d$,

$$
\begin{aligned}
E[e^{\mathrm{i}u^\top X_t}] &= E\Big[\sum_{n\geq 0} \mathbb{1}_{\{N_t=n\}} \prod_{j=1}^n e^{\mathrm{i}u^\top Y_j}\Big] = \sum_{n\geq 0} P[N_t=n] \Big(E[e^{\mathrm{i}u^\top Y_1}]\Big)^n \\
&= \sum_{n\geq 0} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \Big(E[e^{\mathrm{i}u^\top Y_1}]\Big)^n = e^{-\lambda t} \exp\Big(\lambda t E[e^{\mathrm{i}u^\top Y_1}]\Big) \\
&= \exp\Big(\lambda t\Big(E[e^{\mathrm{i}u^\top Y_1}] - 1\Big)\Big).
\end{aligned}
$$

Let $\nu^Y$ be the distribution of $Y_1$ and $\nu := \lambda\nu^Y$. Then

$$
\lambda\Big(E[e^{\mathrm{i}u^\top Y_1}] - 1\Big) = \lambda \int (e^{\mathrm{i}u^\top x} - 1) d\nu^Y = \int (e^{\mathrm{i}u^\top x} - 1) d\nu.
$$

Truncating as in the lecture notes, we can decompose

$$
E[e^{\mathrm{i}u^\top X_t}] = \exp\left(t\int (e^{\mathrm{i}u^\top x} - 1)d\nu\right) = \exp\left(t\left(\int x\mathbb{1}_{|x|\leq 1}\,d\nu + \int (e^{\mathrm{i}u^\top x} - 1 - x\mathbb{1}_{|x|\leq 1})d\nu\right)\right).
$$

Therefore, we obtain the triplet $(b, 0, \nu)$, where $b = \int_{\{x\colon |x|\leq 1\}} x\,d\nu$.

(b) The characteristic function of $X_1$ is

$$
f_1(u) = \varphi_{X_1}(u) = \int_0^1 e^{\mathrm{i}ux}dx = \left[\frac{e^{\mathrm{i}ux}}{\mathrm{i}u}\right]_{x=0}^{x=1} = \frac{e^{\mathrm{i}u} - 1}{\mathrm{i}u},
$$

which has a zero at $u = 2\pi$. This would contradict Exercise **13.1 (d)**, hence there is no such Lévy process.

**Alternative proof:** We can generalise the result to any random variable $X_1$ with compact support $\operatorname{supp}(X_1) \subseteq [a, b]$, for some $a < b$. We claim that if $X_1$ is infinitely divisible, then $X_1$ is constant. Hence there is no Lévy process $X$ such that $X_1$ is uniformly distributed on $[0, 1]$. By infinite divisibility, for each $n$ we have $X_1 = \sum_{j=1}^n Y_j^n$, where the random variables $(Y_j^n)_{j=1}^n$ are i.i.d. This implies that $\operatorname{supp}(Y_j^n) \subseteq [a/n, b/n]$. Indeed, suppose by contradiction that $P\big[Y_j^n > b/n\big] > 0$. As the $Y_j^n$ are i.i.d., we would have that

$$
P\big[X_1 > b\big] \geq P\left[\bigcap_{j=1}^n \left\{Y_j^n > \frac{b}{n}\right\}\right] = P\left[Y_j^n > \frac{b}{n}\right]^n > 0,
$$

which contradicts the fact that $\operatorname{supp}(X_1) \subseteq [a, b]$. The case $P\big[Y_j^n < a/n\big] > 0$ is analogous. Since $\operatorname{supp}(Y_j^n) \subseteq [a/n, b/n]$, we have that $\operatorname{Var}(Y_j^n) \leq (b-a)^2/n^2$. Therefore, we can bound $\operatorname{Var}(X_1) \leq (b-a)^2/n \to 0$ as $n \to \infty$, so $X_1$ is constant.

(c) We need to show that $(X_{t_1}, \dots, X_{t_n})$ is independent of $(Y_{t_1}, \dots, Y_{t_n})$ for any $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_n$. This follows if we can show that the random variables

$$
X_{t_1} - X_{t_0},\, \dots,\, X_{t_n} - X_{t_{n-1}},\, Y_{t_1} - Y_{t_0},\, \dots,\, Y_{t_n} - Y_{t_{n-1}}
$$

are independent. For $j = 1, \dots, n$ and $u_j, v_j \in \mathbb{R}^d$, we have that

$$
\begin{aligned}
E\left[\prod_{j=1}^n e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})}\right] &= E\left[E\left[\prod_{j=1}^n e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})} \,\middle|\, \mathcal{F}_{t_{n-1}}\right]\right] \\
&= E\left[E\left[e^{\mathrm{i}u_j^\top (X_{t_n} - X_{t_{n-1}}) + \mathrm{i}v_j^\top (Y_{t_n} - Y_{t_{n-1}})} \,\middle|\, \mathcal{F}_{t_{n-1}}\right] \prod_{j=1}^{n-1} e^{\mathrm{i}u_j^\top (X_{t_j} - X_{t_{j-1}}) + \mathrm{i}v_j^\top (Y_{t_j} - Y_{t_{j-1}})}\right].
\end{aligned}
$$

Since $(X,Y)$ is a Lévy process with respect to $\mathbb{F}$, in particular so is $u_j^\top X + v_j^\top Y$, so that $u_j^\top(X_{t_n} - X_{t_{n-1}}) + v_j^\top(Y_{t_n} - Y_{t_{n-1}})$ is independent of $\mathcal{F}_{t_{n-1}}$ and has the same distribution as $u_j^\top X_{t_n - t_{n-1}} + v_j^\top Y_{t_n - t_{n-1}}$. Therefore, the expression above is equal to

$$E\left[e^{\mathrm{i}u_j^\top X_{t_n-t_{n-1}} + \mathrm{i}v_j^\top Y_{t_n-t_{n-1}}}\right] E\left[\prod_{j=1}^{n-1} e^{\mathrm{i}u_j^\top(X_{t_j}-X_{t_{j-1}}) + \mathrm{i}v_j^\top(Y_{t_j}-Y_{t_{j-1}})}\right].$$

We can apply an inductive argument to the remaining product to obtain that

$$E\left[\prod_{j=1}^{n} e^{\mathrm{i}u_j^\top(X_{t_j}-X_{t_{j-1}}) + \mathrm{i}v_j^\top(Y_{t_j}-Y_{t_{j-1}})}\right] = \prod_{j=1}^{n} E\left[e^{\mathrm{i}u_j^\top X_{t_j-t_{j-1}} + \mathrm{i}v_j^\top Y_{t_j-t_{j-1}}}\right].$$

Finally, by the assumption on $X$ and $Y$ we have that

$$\begin{aligned}\prod_{j=1}^{n} E\left[e^{\mathrm{i}u_j^\top X_{t_j-t_{j-1}} + \mathrm{i}v_j^\top Y_{t_j-t_{j-1}}}\right] &= \prod_{j=1}^{n} E\left[e^{\mathrm{i}u_j^\top X_{t_j-t_{j-1}}}\right] E\left[e^{\mathrm{i}v_j^\top Y_{t_j-t_{j-1}}}\right] \\ &= \prod_{j=1}^{n} E\left[e^{\mathrm{i}u_j^\top(X_{t_j}-X_{t_{j-1}})}\right] E\left[e^{\mathrm{i}v_j^\top(Y_{t_j}-Y_{t_{j-1}})}\right].\end{aligned}$$

This shows our claim that the random variables

$$X_{t_1} - X_{t_0},\, \dots,\, X_{t_n} - X_{t_{n-1}},\, Y_{t_1} - Y_{t_0},\, \dots,\, Y_{t_n} - Y_{t_{n-1}}$$

are independent. In particular, the vectors

$$\big(X_{t_1} - X_{t_0},\, \dots,\, X_{t_n} - X_{t_{n-1}}\big),\, \big(Y_{t_1} - Y_{t_0},\, \dots,\, Y_{t_n} - Y_{t_{n-1}}\big)$$

are independent, and so are $(X_{t_1}, \dots, X_{t_n})$ and $(Y_{t_1}, \dots, Y_{t_n})$. Since $0 = t_0 < t_1 < \cdots < t_n$ are arbitrary, this shows that $X$ and $Y$ are independent.

**Exercise 13.3**

(a) Let $\tilde{\nu}$ be a finite measure supported on $[\varepsilon, \infty)$ for some $\varepsilon > 0$, and $\tilde{\lambda} := \tilde{\nu}([\varepsilon, \infty)) > 0$. Suppose that $(N_t)$ is a Poisson process with rate $\tilde{\lambda}$ and $(\tilde{Y}_n)$ are i.i.d. random variables with distribution $\tilde{\lambda}^{-1}\tilde{\nu}$. Check using Exercise **13.2(a)** that the process $J_t^{\tilde{\nu}} := \sum_{j=1}^{N_t} \tilde{Y}_j$ is a Lévy process with Lévy triplet $(\tilde{b}, 0, \tilde{\nu})$, where $\tilde{b} = \int \mathbb{1}_{\{|x|\le 1\}} x \, d\tilde{\nu}$.

(b) Suppose that $\tilde{\nu}$ has compact support, i.e., $\tilde{\nu}((K, \infty)) = 0$ for some $K \in (0, \infty)$, so that $\tilde{\nu}([\varepsilon, K]) = \tilde{\lambda}$. Find a constant $\mu > 0$ such that the process $M^{\tilde{\nu}}$ defined by

$$M_t^{\tilde{\nu}} := J_t^{\tilde{\nu}} - \mu t$$

is a martingale. If $\tilde{\nu}$ is not compactly supported, under what assumption can we find such a constant $\mu$?

(c) For some $K > 0$, let $\nu$ be a measure supported on $[0, K]$ such that $\nu(\{0\}) = 0$ and $\nu((\varepsilon, K]) < \infty$ for each $\varepsilon > 0$. Choose a sequence $(a_m)_{m\in\mathbb{N}_0}$ such that $a_0 = K$ and $a_m \searrow 0$, and let $(\nu_m)_{m\in\mathbb{N}}$ be a sequence of measures that are absolutely continuous with respect to $\nu$ with respective densities $\frac{d\nu_m}{d\nu} = \mathbb{1}_{(a_m, a_{m-1}]}$. As in (a), for each $m \in \mathbb{N}$, let $(N_t^m)_{t\ge 0}$ be a Poisson process with rate $C_m := \nu((a_m, a_{m-1}])$ and $(Y_n^m)$ be i.i.d. random variables with distribution $C_m^{-1}\nu_m$. We suppose that the $(N^m)$ and $(Y_n^m)$ are all independent, and define $J^{\nu_m}$ and $M^{\nu_m}$ as in (a) and (b).

Show that for each $k \ge 1$, the process $J^k := \sum_{m=1}^k J^{\nu_m}$ is Lévy and find its Lévy triplet. Find a constant $\mu_k$ such that $M_t^k := J_t^k - \mu_k t$ is a martingale.

(d) Suppose that $\int_0^K x^2 \nu(dx) < \infty$. For any $T > 0$, show that the sequence of stopped martingales $((M^k)^T)_{k\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$.

(e) Under the assumption in (d), does $(J^k)_{k\in\mathbb{N}}$ converge?

**Solution 13.3**

(a) This is an immediate check from Exercise **13.2(a)**. Note that $J^{\tilde{\nu}}$ is a compound Poisson process, where the jumps $\tilde{Y}_n$ have distribution $F = \tilde{\lambda}^{-1}\tilde{\nu}$. Therefore, $J^{\tilde{\nu}}$ is a Lévy process with triplet $(\tilde{b}, 0, \tilde{\lambda}\tilde{\lambda}^{-1}\tilde{\nu}) = (\tilde{b}, 0, \tilde{\nu})$, as we wanted.

(b) Note that $J_t^{\tilde{\nu}} - \mu t$ is a Lévy process for any $\mu > 0$. Moreover, $J_1^{\tilde{\nu}} \in L^1$ because $J^{\tilde{\nu}}$ is a compound Poisson process and $\tilde{Y}_1 \in L^1$, since it has compact support. Therefore, it is enough to check that $E[J_1^{\tilde{\nu}} - \mu] = 0$. Indeed, we have by independence that

$$E[J_1^{\tilde{\nu}}] = E\left[\sum_{n=1}^{N_1} Y_n\right] = E\left[E\left[\sum_{n=1}^{\hat{n}} Y_n\right]_{\hat{n}=N_1}\right] = E\left[N_1 \tilde{\lambda}^{-1} \int_\varepsilon^K x\tilde{\nu}(dx)\right] = \int_\varepsilon^K x\tilde{\nu}(dx).$$

For $\mu = \int_\varepsilon^K x\tilde{\nu}(dx)$, we thus have that $J_1^{\tilde{\nu}} - \mu$ is integrable with $E[J_1^{\tilde{\nu}} - \mu] = 0$. Therefore, $M^{\tilde{\nu}}$ is a martingale by Proposition 5.2.2 of the notes. If $\tilde{\nu}$ is not compactly supported, we can still find $\mu = \int_\varepsilon^\infty x d\tilde{\nu}(x)$ by the same argument, as long as the integral is finite.

(c) It follows immediately from (a) that each $J^{\nu_m}$ is a Lévy process with triplet $(b_m, 0, \nu_m)$. Moreover, the $(J^{\nu_m})_{m\in\mathbb{N}}$ are independent, by construction. One can easily that the sum of independent Lévy processes $J^k := \sum_{m=1}^k J^{\nu_m}$ is also Lévy. Moreover, $J^k$ has the Lévy triplet $(\hat{b}_k, 0, \sum_{m=1}^k \nu_m) = (\hat{b}_k, 0, \hat{\nu}_k)$, where $\hat{\nu}_k$ has density $\frac{d\hat{\nu}_k}{d\nu} = \mathbb{1}_{(a_k, K]}$ and $\hat{b}_k = \int \mathbb{1}_{\{|x|\le 1\}} x \, d\hat{\nu}_k$. As in (b), we have that $\mu_k$ is given by

$$\mu_k = \int x\hat{\nu}_k(dx) = \int \mathbb{1}_{\{x\in(a_m, K]\}} x\nu(dx).$$

(d) We show that $(M^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_0^2$. Note that $\|M^T\|_{\mathcal{H}_0^2} = E\left[[M]_T\right]$ and each $M^k$ has finite variation. Also, by independence, the probability of two of the processes $J^{\nu_m}, J^{\nu_{m'}}$ jumping simultaneously is 0. Therefore, for $k' \geq k$, we can compute

$$\begin{aligned}
E\left[[M^{k'} - M^k]_T\right] &= E\left[\sum_{0<s\leq T} \Delta(M^{k'} - M^k)_s^2\right] = E\left[\sum_{0<s\leq T}\sum_{m=k+1}^{k'} \Delta(J^{\nu_m})_s^2\right] \\
&= E\left[\sum_{m=k+1}^{k'}\sum_{n=1}^{N_T^m}(Y_n^m)^2\right] = \sum_{m=k+1}^{k'} E\left[E\left[\sum_{n=1}^{\hat{n}}(Y_n^m)^2\right]_{\hat{n}=N_T^m}\right] \\
&= \sum_{m=k+1}^{k'} E\left[N_T^m(\lambda_m)^{-1}\int \mathbb{1}_{\{x\in(a_m,a_{m-1}]\}}x^2\nu(dx)\right] \\
&= \sum_{m=k+1}^{k'} \int \mathbb{1}_{\{x\in(a_m,a_{m-1}]\}}x^2\nu(dx) \\
&= \int \mathbb{1}_{\{x\in(a_{k'},a_k]\}}x^2 d\nu(x) \leq \int \mathbb{1}_{\{x\in[0,a_k]\}}x^2 d\nu(x).
\end{aligned}$$

Since we assume that $\int_0^K x^2\nu(dx) < \infty$, it follows that $\lim_{k\to\infty}\sup_{k'\geq k} E\left[[M^{k'} - M^k]_T\right] = 0$ by the dominated convergence theorem. Therefore, $((M^k)^T)$ is a Cauchy sequence in $\mathcal{H}_0^2$ and thus it converges.

(e) In general, it is not the case that $(J^k)_{k\in\mathbb{N}}$ converges. For example, let $\nu$ have density with respect to Lebesgue measure $\frac{d\nu(x)}{dx} = \mathbb{1}_{\{x\in(0,1]\}}x^{-2}$. If $a_m = e^{-m}$, we have that

$$\mu_k = \int_{e^{-k}}^1 x^{-1}dx = [\log x]_{x=e^{-k}}^{x=1} = k \to \infty.$$

Since $((M^k)^1)$ converges in $\mathcal{H}_0^2$, in particular $(M_1^k)$ converges in probability. Since $(\mu_k)$ diverges, it follows that $(J^k)$ cannot converge.