Brownian Motion and Stochastic Calculus

Exercise sheet 2

Exercise 2.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space, W a Brownian motion on $[0, \infty)$, Z a random variable independent of W and $t^* \in (0, \infty)$. We define another stochastic process $W' = (W'_t)_{t \geq 0}$ by

$$W'_t = W_t 1_{\{t < t^*\}} + \Big(W_{t^*} + Z(W_t - W_{t^*})\Big) 1_{\{t \geq t^*\}}.$$

Find all possible distributions of Z such that W' is a Brownian motion.

Solution 2.1 We claim that W' is a Brownian motion if and only if Z takes values in $\{-1, 1\}$ P-almost surely. It is clear that $P[W'_0 = 0] = 1$ and that W' is P-a.s. continuous. It only remains to prove that it has Gaussian independent increments with the correct variance.

To that end, take $0 \leq t_0 < \cdots < t_k \leq t^* < t_{k+1} < \cdots < t_n$ and note that the characteristic function $\varphi_V(\lambda_1, \ldots, \lambda_n)$ of the random vector $V := (W'_{t_0} - W'_{t_1}, \ldots, W'_{t_n} - W'_{t_{n-1}})$ is given by

$$\begin{aligned}
&E\left[\exp\left(i \sum_{j=1}^n \lambda_j (W'_{t_j} - W'_{t_{j-1}})\right)\right] \\
&= E\left[e^{i\lambda_{k+1}(W'_{t^*} - W'_{t_k}) + \sum_{j=1}^k i\lambda_j (W'_{t_j} - W'_{t_{j-1}})}\right] E\left[e^{iZ\left(\lambda_{k+1}(W_{t_{k+1}} - W_{t^*}) + \sum_{j=k+2}^n \lambda_i (W_{t_j} - W_{t_{j-1}})\right)}\right] \\
&= \exp\left(-\frac{1}{2}\lambda_{k+1}^2 (t^* - t_k) - \frac{1}{2}\sum_{j=1}^k \lambda_j^2 (t_j - t_{j-1})\right) E\left[e^{iZ\left(\lambda_{k+1}(W_{t_{k+1}} - W_{t^*}) + \sum_{j=k+2}^n \lambda_j (W_{t_j} - W_{t_{j-1}})\right)}\right],
\end{aligned}$$

where we have used the independence of the increments of Brownian motion and Z, as well as that the characteristic function of a centred normal random variable with variance σ^2 is $\varphi(\lambda) = \exp(-\lambda^2\sigma^2/2)$. For the second expectation, note that

$$\begin{aligned}
&E\left[e^{iZ\left(\lambda_{k+1}(W_{t_{k+1}} - W_{t^*}) + \sum_{j=k+2}^n \lambda_j (W_{t_j} - W_{t_{j-1}})\right)}\right] \\
&= E\left[E\left[e^{iZ\lambda_{k+1}(W_{t_{k+1}} - W_{t^*}) + iZ\sum_{j=k+2}^n \lambda_j (W_{t_j} - W_{t_{j-1}})} \,\Big|\, Z\right]\right] \\
&= E\left[\exp\bigg(-\frac{1}{2} Z^2 \bigg[\lambda_{k+1}^2 (t_{k+1} - t^*) + \sum_{j=k+2}^n \lambda_j^2 (t_j - t_{j-1})\bigg]\bigg)\right].
\end{aligned}$$

We conclude that we can obtain

$$\forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \qquad \varphi_V(\lambda_1, ..., \lambda_n) = \exp\left(-\frac{1}{2}\sum_{j=1}^n \lambda_j^2 (t_j - t_{j-1})\right)$$

which is the characteristic function of centred independent normal variables with the required variance, if and only if Z takes values in $\{-1, 1\}$ P-a.s. The sufficiency follows immediately, while the necessity can be shown using the fact that the Laplace transform of Z^2, given by the function $\rho \mapsto E[e^{-\rho Z^2}]$ on $[0, \infty)$, is unique and therefore $Z^2 = 1$ almost surely.

Exercise 2.2 Let X be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with $X_0 = 0$ P-a.s., and let $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ denote the (raw) filtration generated by X, i.e., $\mathcal{F}_t^X = \sigma(X_s;\, s \leq t)$. Show that the following two properties are equivalent:

(i) X has *independent increments*, i.e., for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the increments $X_{t_i} - X_{t_{i-1}}$, $i = 1, \ldots, n$, are independent.

(ii) X has $\mathbb{F}^X$*-independent increments*, i.e., $X_t - X_s$ is independent of $\mathcal{F}_s^X$ whenever $t \geq s$.

Remark: This also shows the equivalence between the two definitions of Brownian motion with properties (BM2) and (BM2′), respectively, when we choose $\mathbb{F} = \mathbb{F}^W$.

Hint: For proving "(i) ⇒ (ii)", you can use the monotone class theorem. When choosing $\mathcal{H}$, recall that a random variable Y is independent of a σ-algebra $\mathcal{G}$ if and only if one has the product formula $E[g(Y)Z] = E[g(Y)]E[Z]$ for all bounded Borel-measurable functions $g : \mathbb{R} \to \mathbb{R}$ and all bounded $\mathcal{G}$-measurable random variables Z.

Solution 2.2 First, assume that X has independent increments and fix $0 \leq s \leq t$. The family

$$\mathcal{M}_s = \left\{ \prod_{i=1}^n h_i(X_{s_i}) : n \in \mathbb{N}, 0 \leq s_1 < \cdots < s_n \leq s, h_i : \mathbb{R} \to \mathbb{R} \text{ Borel and bounded} \right\}$$

of bounded, real-valued functions on Ω is closed under multiplication. Moreover, note that $\sigma(\mathcal{M}_s) = \mathcal{F}_s^X$. Let $\mathcal{H}_s$ denote the real vector space of all bounded, real-valued, $\mathcal{F}_s^X$-measurable functions Z on Ω with the property that:

$$E[g(X_t - X_s)Z] = E[g(X_t - X_s)]E[Z] \quad \text{for all bounded Borel functions } g : \mathbb{R} \to \mathbb{R}.$$

Clearly, $\mathcal{H}_s$ contains the constant function 1 and is closed under monotone bounded convergence (we even do not need monotonicity).

Next, we show that $\mathcal{H}_s$ contains $\mathcal{M}_s$. Fix a typical element $Z = \prod_{i=1}^n h_i(X_{s_i})$ in $\mathcal{M}_s$. Define the function $h : \mathbb{R}^n \to \mathbb{R}$ by $h(x) = \prod_{i=1}^n h_i(x_i)$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Note that h is again a Borel function. Then we can write $Z = h(X_{s_1}, \ldots, X_{s_n})$. Since $X_0 = 0$ P-a.s., we also have

$$(X_{s_1}, \ldots, X_{s_n}) = f(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}}) \quad P\text{-a.s.}$$

for a linear transformation f. Finally,

$$\begin{aligned}
E[g(X_t - X_s)Z] &= E[g(X_t - X_s)h(X_{s_1}, \ldots, X_{s_n})] \\
&= E[g(X_t - X_s)(h \circ f)(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})] \\
&= E[g(X_t - X_s)]E[(h \circ f)(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})] \\
&= E[g(X_t - X_s)]E[Z]
\end{aligned}$$

where we use the assumption that $X_t - X_s$ is independent of $(X_{s_1} - X_0, X_{s_2} - X_{s_1}, \ldots, X_{s_n} - X_{s_{n-1}})$ in the third equality. Thus, $Z \in \mathcal{H}_s$.

The monotone class theorem yields that $\mathcal{H}_s$ contains every bounded $\mathcal{F}_s^X$-measurable function on Ω. In particular, $X_t - X_s$ is independent of $\mathcal{F}_s^X$.

For the converse implication, we proceed by induction on n. The case $n = 1$ is trivial, so fix $n \geq 2$, $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, and $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, n$. Conditioning on $\mathcal{F}_{t_{n-1}}^X$ and using (ii) for $t = t_n$ and $s = t_{n-1}$, we obtain

$$P\left[\bigcap_{i=1}^n (X_{t_i} - X_{t_{i-1}})^{-1}(A_i)\right] = P\left[\bigcap_{i=1}^{n-1} (X_{t_i} - X_{t_{i-1}})^{-1}(A_i)\right] P\left[(X_{t_n} - X_{t_{n-1}})^{-1}(A_n)\right].$$

Applying the induction hypothesis to the first factor on the right-hand side completes the proof.

Exercise 2.3 A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder-continuous of order α at $x \in D$ if there exist $\delta > 0$ and $C > 0$ such that $|f(x) - f(y)| \le C|x - y|^\alpha$ for all $y \in D$ with $|x - y| \le \delta$. A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is called locally Hölder-continuous of order α if it is locally Hölder-continuous of order α at each $x \in D$.

(a) Let $Z \sim N(0, 1)$. Prove that $P[|Z| \le \varepsilon] \le \varepsilon$ for any $\varepsilon \ge 0$.

(b) Prove that for any $\alpha > \frac{1}{2}$, P-almost all Brownian paths are nowhere on $[0, 1]$ locally Hölder-continuous of order α.
Hint: Take any $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$ and show that the set $\{W_\cdot(\omega)$ is locally α-Hölder at some $s \in [0, 1]\}$ is contained in the set $\bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^M \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \right\}$.

(c) The *Kolmogorov–Čentsov theorem* states that an $\mathbb{R}$-valued process X on $[0, T]$ satisfying

$$E\big[|X_t - X_s|^\gamma\big] \le C\,|t - s|^{1+\beta}, \quad s, t \in [0, T],$$

where $\gamma, \beta, C > 0$, has a version which is locally Hölder-continuous of order α for all $\alpha < \beta/\gamma$. Use this to deduce that Brownian motion is for every $\alpha < 1/2$ locally Hölder-continuous of order α.

Remark: One can also show that the Brownian paths are *not* locally Hölder-continuous of order 1/2. The exact modulus of continuity was found by P. Lévy.

Solution 2.3

(a) The density $f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ of Z is bounded by $\frac{1}{\sqrt{2\pi}} \le \frac{1}{2}$. So

$$P[|Z| \le \varepsilon] = P[-\varepsilon \le Z \le \varepsilon] = \int_{-\varepsilon}^{\varepsilon} f(x)\,dx \le \frac{1}{2}2\varepsilon = \varepsilon.$$

(b) Take any $\alpha > \frac{1}{2}$ and let $M \in \mathbb{N}$ satisfying $M(\alpha - \frac{1}{2}) > 1$. If $W_\cdot(\omega)$ is locally Hölder-continuous of order α at the point $s \in [0, 1]$, there exists a constant C so that $|W_t(\omega) - W_s(\omega)| \le C|t - s|^\alpha$ for t near s. Then $|W_{\frac{k}{n}}(\omega) - W_{\frac{k-1}{n}}(\omega)| \le Cn^{-\alpha}$ for all large enough n, for $\frac{k}{n}$ near s and M successive indices k. The set $\{W_\cdot(\omega)$ is locally α-Hölder at some $s \in [0, 1]\}$ is therefore contained in

$$B := \bigcup_{C\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{n\ge m} \bigcup_{k=0,\dots,n-M} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \right\}.$$

We show that this is a nullset. As the above Brownian increments are i.i.d $\sim N(0, \frac{1}{n})$, and using (a) for $Z \sim \mathcal{N}(0, 1)$, we have

$$P\left[\bigcap_{i=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \right\}\right] = \left(P\left[|Z| \le \frac{C}{n^{\alpha-1/2}}\right]\right)^M \le C^M n^{-M(\alpha-\frac{1}{2})}. \tag{1}$$

Now, we have

$$\begin{aligned} D_m :&= \bigcap_{n\ge m} \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \right\} \\ &\subseteq \bigcup_{k=0,\dots,n-1} \bigcap_{j=1}^{M} \left\{ |W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \le C\frac{1}{n^\alpha} \right\} \quad \text{for each } n \ge m \end{aligned}$$

and therefore, due to (1), as $M(\alpha - \frac{1}{2}) > 1$, we get

$$\begin{aligned} P\big[D_m\big] &\leq \limsup_{n\to\infty} P\left[\bigcup_{k=0,\dots,n-M}\bigcap_{j=1}^{M}\left\{|W_{\frac{k+j}{n}}(\omega) - W_{\frac{k+j-1}{n}}(\omega)| \leq C\frac{1}{n^\alpha}\right\}\right] \\ &\leq \limsup_{n\to\infty} n\, C^M\, n^{-M(\alpha-\frac{1}{2})} \\ &= 0. \end{aligned}$$

Therefore, since B is a countable union of nullsets, $P[B] = 0$.

(c) Let $Y_\sigma \sim \mathcal{N}(0,\sigma^2)$ for any $\sigma \geq 0$. We note that $E[Y_\sigma^m] = C_m\sigma^m$, where $C_m = E[Y_1^m]$. Thus

$$E\big[|W_t - W_s|^{2n}\big] = C_{2n}|t-s|^n \quad \text{for all } n.$$

Writing $\gamma_n := 2n$ and $\beta_n := n-1$ yields that

$$E\big[|W_t - W_s|^{\gamma_n}\big] = C_{2n}|t-s|^{1+\beta_n} \quad \text{for all } n.$$

Now, fix $\alpha < \frac{1}{2}$. As $\frac{\beta_n}{\gamma_n} < \frac{1}{2}$ for any $n \in \mathbb{N}$ and $\frac{\beta_n}{\gamma_n}$ converges to $\frac{1}{2}$, we find big enough N such that $\alpha < \frac{\beta_N}{\gamma_N}$. Thus, we get that W has a locally α-Hölder continuous version by the *Kolmogorov–Čentsov theorem.*

However, note that both W and this version are continuous, and therefore by exercise **1.2**, they are indistinguishable. Therefore, W itself is locally α-Hölder continuous.

Remark: In fact, $E[Y_\sigma^n] = (n-1)!!\,\sigma^n$ for n even and 0 otherwise, where $n!!$ denotes the double factorial, defined as the product of every odd number between n and 1.

Exercise 2.4

(a) Let W be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_t = \sigma(W_s, 0 \le s \le t)$ be the natural filtration of W. Let $\mathcal{F}_{0+} := \cap_{t>0}\mathcal{F}_t$. Show *Blumenthal's 0-1 law*: for $A \in \mathcal{F}_{0+}$, either $P[A] = 0$ or $P[A] = 1$.

Hint: Show that A and the increments of W are independent.

(b) Show that

$$P\left[\limsup_{t\searrow 0} \frac{W_t}{\sqrt{t}} = \infty\right] = 1.$$

Hint: Start by showing that for each $C > 0$,

$$\lim_{t\searrow 0} P\left[\sup_{0\le s\le t} \left(W_s - C\sqrt{s}\right) > 0\right] > 0$$

and use (a).

Solution 2.4

(a) By construction, we note that $\mathcal{F}_{0+} \subseteq \mathcal{F}_t$ for any $t > 0$. Letting $\mathcal{G}_t := \sigma(W_u - W_t, u \ge t)$, and since $\mathcal{F}_t$ and $\mathcal{G}_t$ are independent due to the independence of increments, it follows that $\mathcal{F}_{0+}$ is independent of $\mathcal{G}_t$ for all $t > 0$. By Dynkin's lemma, it follows that $\mathcal{F}_{0+}$ is independent of

$$\mathcal{G}_{0+} = \sigma\left(\bigcup_{t>0} \mathcal{G}_t\right) = \sigma(W_t - W_u; t \ge u > 0),$$

taking $\bigcup_{t>0} \mathcal{G}_t$ as a π-system.

We claim that

$$\mathcal{F}_\infty = \sigma(W_t; t \ge 0) \subseteq \overline{\mathcal{G}_{0+}},$$

where $\overline{\mathcal{G}_{0+}} = \sigma(\mathcal{G}_{0+}, \mathcal{N}\})$ denotes the completion by P-nullsets. It suffices to show that for any $t \ge 0$ and Borel $B \subseteq \mathbb{R}$,

$$\{W_t \in B\} \in \overline{\mathcal{G}_{0+}}.$$

Indeed, denoting by "N_i" any nullset, we have that

$$\begin{aligned}
\{W_t \in B\} &= \{W_t \in B, W \text{ is continuous}, W_0 = 0\} \cup N_1 \\
&= \left\{\lim_{u\searrow 0, u\in\mathbb{Q}} (W_t - W_u) \in B, W \text{ is continuous}, W_0 = 0\right\} \cup N_1 \\
&= \left(\left\{\lim_{u\searrow 0, u\in\mathbb{Q}} (W_t - W_u) \in B\right\} \cap N_2^c\right) \cup N_1 \in \overline{\mathcal{G}_{0+}}.
\end{aligned}$$

It follows that for any $A \in \mathcal{F}_{0+} \subseteq \mathcal{F}_\infty$, we have that $A = \tilde{A} \cup N$ for some $\tilde{A} \in \mathcal{G}_{0+}$ and a nullset N. Therefore, we can use independence to show that

$$P[A] = P[A \cap \tilde{A}] = P[A]P[\tilde{A}] = P[A]^2$$

and therefore $P[A] \in \{0, 1\}$.

(b) Let $C > 0$. For any $t > 0$, we have that $W_t \sim \mathcal{N}(0, t)$ and therefore

$$P[W_t > C\sqrt{t}] = 1 - \Phi(C),$$

where Φ is the cdf of the standard normal distribution. In particular, we have that

$$\lim_{t\searrow 0} P\left[\sup_{0\le s\le t}\left(W_s - C\sqrt{s}\right) > 0\right] \ge \lim_{t\searrow 0} P\left[W_t - C\sqrt{t} > 0\right] = 1-\Phi(C) > 0.$$

We deduce that

$$P\left[\limsup_{t\searrow 0}\frac{W_t}{\sqrt{t}} \ge C\right] = P\left[\forall t\in\mathbb{Q}_{++},\ \sup_{0\le s\le t}\frac{W_s}{\sqrt{s}}\ge C\right] = \inf_{t\in\mathbb{Q}_{++}} P\left[\sup_{0\le s\le t}\frac{W_s}{\sqrt{s}}\ge C\right] \ge 1-\Phi(C) > 0,$$

using the monotone convergence theorem. Noting that $\left\{\limsup_{t\searrow 0}\frac{W_t}{\sqrt{t}}\ge C\right\}\in\overline{\mathcal{F}_{0+}}$, it follows by (a) that this probability is equal to 1. We conclude that

$$P\left[\limsup_{t\searrow 0}\frac{W_t}{\sqrt{t}} = \infty\right] = \inf_{C\in\mathbb{Q}_{++}} P\left[\limsup_{t\to 0}\frac{W_t}{\sqrt{t}}\ge C\right] = 1.$$