Brownian Motion and Stochastic Calculus

Exercise sheet 3

Exercise 3.1 Given a measurable space $(\Omega, \mathcal{F})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, we set $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$ and define for any $\mathbb{F}$-stopping time τ the σ-field

$$\mathcal{F}_\tau := \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \text{ for all } t \geq 0\}.$$

Let S, T be two $\mathbb{F}$-stopping times. Show that:

(a) if $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$, and in general, $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

(b) $\{S < T\}$, $\{S \leq T\}$ belong to $\mathcal{F}_S \cap \mathcal{F}_T$. Moreover, for any $A \in \mathcal{F}_S$, $A \cap \{S < T\}$ and $A \cap \{S \leq T\}$ belong to $\mathcal{F}_{S\wedge T}$.

(c) For any stopping time τ,

$$\mathcal{F}_\tau = \sigma(X_\tau : X \text{ an optional process}).$$

Solution 3.1

(a) Suppose that $S \leq T$ and let $A \in \mathcal{F}_S$. For $t \geq 0$, we have $\{T \leq t\} \subseteq \{S \leq t\}$ so that

$$A \cap \{T \leq t\} = (A \cap \{S \leq t\}) \cap \{T \leq t\}.$$

But $A \cap \{S \leq t\} \in \mathcal{F}_t$ as $A \in \mathcal{F}_S$, while $\{T \leq t\} \in \mathcal{F}_t$ as T is a stopping time. Thus, $A \cap \{T \leq t\} \in \mathcal{F}_t$ and so $A \in \mathcal{F}_T$. This shows the first part.

For the second part, note that $S \wedge T \leq S$ and $S \wedge T \leq T$, so we immediately have that $\mathcal{F}_{S\wedge T} \subseteq \mathcal{F}_S \cap \mathcal{F}_T$. To show the reverse inclusion, let $A \in \mathcal{F}_S \cap \mathcal{F}_T$. We have that

$$A \cap \{S \wedge T \leq t\} = (A \cap \{S \leq t\}) \cup (A \cap \{T \leq t\}) \in \mathcal{F}_t,$$

since $A \cap \{S \leq t\}, A \cap \{T \leq t\} \in \mathcal{F}_t$, which shows the result.

(b) We have that, for $t \geq 0$,

$$\{S < T\} \cap \{T \leq t\} = \bigcup_{q \in \mathbb{Q}\cap[0,t]} \big(\{S \leq q\} \cap \{q < T\} \cap \{T \leq t\}\big) \in \mathcal{F}_t$$

since each of these events is in $\mathcal{F}_t$. Likewise,

$$\{S < T\} \cap \{S \leq t\} = \big(\{S \leq t\} \cap \{t < T\}\big) \cup \big(\{S < T\} \cap \{T \leq t\}\big) \in \mathcal{F}_t.$$

This holds for each $t \geq 0$, so $\{S < T\} \in \mathcal{F}_S \cap \mathcal{F}_T = \mathcal{F}_{S\wedge T}$. We also have that $\{S \leq T\} = \{T < S\}^c \in \mathcal{F}_S \cap \mathcal{F}_T$, by symmetry.

Given $A \in \mathcal{F}_S$, for any $t \geq 0$,

$$A \cap \{S < T\} \cap \{S \wedge T \leq t\} = (A \cap \{S \leq t\}) \cap (\{S < T\} \cap \{S \leq t\}) \in \mathcal{F}_t$$

since $A, \{S < T\} \in \mathcal{F}_S$. Similarly,

$$A \cap \{S \leq T\} \cap \{S \wedge T \leq t\} = (A \cap \{S \leq t\}) \cap (\{S \leq T\} \cap \{S \leq t\}) \in \mathcal{F}_t.$$

(c) Let $A \in \mathcal{F}_\tau$. We can define the process $X = \mathbb{1}_{A\cap\{t\geq\tau\}}$, which we claim is optional. Indeed, it is RCLL with a single jump at time τ, and it is adapted, since for each $t \geq 0$, we have that $X_t^{-1}\{1\} = A \cap \{\tau \leq t\} \in \mathcal{F}_t$. This shows the inclusion ' $\subseteq$ ".

Conversely, let X be an optional process. We first assume that X is right-continuous and adapted. Let $t \geq 0$ and $B \subseteq \mathbb{R}$ be an open set. By right-continuity, on $\{\tau < t\}$ we have that $X_\tau \in B$ if and only if there exists some $\varepsilon > 0$ such that $\inf_{s\in(\tau,\tau+\varepsilon)\cap[0,t]} \operatorname{dist}(X_s, B^c) \geq \varepsilon$ (in particular, $X_s \in B$ for any such s). Taking the complement, we can then write

$$\{X_\tau \in B\} \cap \{\tau \leq t\} =$$
$$\big(\{X_t \in B\} \cap \{\tau = t\}\big) \cup \left(\bigcap_{\varepsilon\in\mathbb{Q}_{++}} \bigcup_{s\in\mathbb{Q}\cap[0,t]} \Big(\{\tau < s\} \cap \{s < \tau + \varepsilon\} \cap \{\operatorname{dist}(X_s, B^c) < \varepsilon\}\Big)\right) \in \mathcal{F}_t.$$

Therefore, X_τ is $\mathcal{F}_\tau$-measurable, as we wanted.

In general, let $\mathcal{M}$ be the set of adapted, bounded and right-continuous processes, and let $\mathcal{H}$ be the set of bounded processes such that X_τ is $\mathcal{F}_\tau$-measurable. It is clear that $\mathcal{M}$ is closed under multiplication and $\sigma(\mathcal{M}) = \mathcal{O}$. Moreover, $\mathcal{H}$ is a vector space such that $1 \in \mathcal{M} \subseteq \mathcal{H}$ (by the previous argument) and it is closed under bounded monotone convergence, since if $X^n \nearrow X$ with X bounded and each $X^n \in \mathcal{H}$, then each X^n_τ is $\mathcal{F}_\tau$-measurable and $X^n_\tau \nearrow X_\tau$ which must also be $\mathcal{F}_\tau$-measurable.

Therefore, by the monotone class theorem we have that $\mathcal{H}$ contains all bounded optional processes, and by a truncation argument (taking $X^n = X\mathbb{1}_{\{|X|\leq n\}}$), we find that X_τ is $\mathcal{F}_\tau$-measurable for any optional process X.

Exercise 3.2 Let $(B_t)_{t\geq 0}$ be a Brownian motion and consider the process X defined by

$$X_t := e^{-t} B_{e^{2t}}, \quad t \in \mathbb{R}.$$

(a) Show that $X_t \sim \mathcal{N}(0,1), \quad \forall t \in \mathbb{R}$.

(b) Show that the process $(X_t)_{t\in\mathbb{R}}$ is *time reversible*, i.e. $(X_t)_{t\geq 0} \overset{(d)}{=} (X_{-t})_{t\geq 0}$.

Hint: Use the time inversion property of Brownian motion, i.e., if W is a Brownian motion, then

$$X_t := \begin{cases} 0, & \text{if } t = 0, \\ tW_{1/t}, & \text{if } t > 0, \end{cases}$$

is also a Brownian motion.

Remark: The process X is called an Ornstein–Uhlenbeck process.

Solution 3.2

(a) Fix any $t \in \mathbb{R}$. Since Brownian motion B is a Gaussian process, we get that X_t is normally distributed. It remains to check its mean and variance: for any $t \in \mathbb{R}$,

$$\begin{aligned} \mathrm{E}[X_t] &= 0, \\ \mathrm{Var}(X_t) &= e^{-2t} e^{2t} = 1. \end{aligned}$$

(b) Fix any $n \in \mathbb{N}$ and any $t_1, t_2, \ldots, t_n \geq 0$. It is enough to check that

$$\big(X_{-t_1}, X_{-t_2}, \ldots, X_{-t_n}\big) \overset{(d)}{=} \big(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\big).$$

From the invariance by time inversion property of Brownian motion (cf. Proposition 1.1 in Section 2.1)), we get that for any $\tilde{t}_1, \ldots, \tilde{t}_n \geq 0$,

$$\big(B_{1/\tilde{t}_1}, B_{1/\tilde{t}_2}, \ldots, B_{1/\tilde{t}_n}\big) \overset{(d)}{=} \big(B_{\tilde{t}_1}/\tilde{t}_1, B_{\tilde{t}_2}/\tilde{t}_2, \ldots, B_{\tilde{t}_n}/\tilde{t}_n\big).$$

Therefore, for $\tilde{t}_i := e^{2t_i}$, $i := 1, \ldots, n$, we get that

$$\begin{aligned} \big(X_{-t_1}, X_{-t_2}, \ldots, X_{-t_n}\big) &= \big(e^{t_1} B_{e^{-2t_1}}, e^{t_2} B_{e^{-2t_2}}, \ldots, e^{t_n} B_{e^{-2t_n}}\big) \\ &\overset{(d)}{=} \big(e^{-t_1} B_{e^{2t_1}}, e^{-t_2} B_{e^{2t_2}}, \ldots, e^{-t_n} B_{e^{2t_n}}\big) \\ &= \big(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\big). \end{aligned}$$

Exercise 3.3 Let W be a Brownian motion with respect to its natural filtration. Show that

$$M_t^{(1)} = e^{t/2} \cos W_t, \qquad M_t^{(2)} = tW_t - \int_0^t W_u du, \qquad M_t^{(3)} = W_t^3 - 3tW_t$$

are martingales.

Hint: You may want to use the formula for the characteristic function of a Gaussian random variable. A trigonometric identity for $\cos(a+b)$ may also be useful; alternatively, you may use that for independent random variables X and Y and if the density f_X exists, we have

$$E\big[g(X,Y) \mid Y\big] = \int_{\mathbb{R}} g(x,Y) f_X(x) dx$$

for any bounded measurable function $g : \mathbb{R}^2 \to \mathbb{R}$.

Solution 3.3 It is clear that $M^{(1)}$ is adapted and each $M_t^{(1)}$ is bounded, hence integrable. To show that $M^{(1)}$ is a martingale, first note that for a normal random variable $Z \sim \mathcal{N}(\mu, \sigma^2)$, we have that

$$\int_{\mathbb{R}} \frac{\cos y}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = E[\cos Z] = \mathrm{Re}\left(E[e^{iZ}]\right) = \mathrm{Re}\left(e^{i\mu - \frac{\sigma^2}{2}}\right) = e^{-\frac{\sigma^2}{2}} \cos \mu.$$

Therefore,

$$\begin{aligned}
E[M_t^{(1)} \mid \mathcal{F}_2] &= e^{t/2} E[\cos(W_s + W_t - W_s) \mid \mathcal{F}_s] \\
&= e^{t/2} \int_{\mathbb{R}} \frac{\cos(W_s + y)}{\sqrt{2\pi(t-s)}} e^{-\frac{y^2}{2(t-s)}} dy \\
&= e^{t/2} \int_{\mathbb{R}} \frac{\cos y}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-W_s)^2}{2(t-s)}} dy \\
&= e^{t/2} e^{-\frac{t-s}{2}} \cos W_s \\
&= e^{s/2} \cos W_s,
\end{aligned}$$

as we wanted.

For $M^{(2)}$, we have that it is adapted and for each t,

$$E[|M_t^{(2)}|] \le tE[|W_t|] + \int_0^t E[|W_u|] du < \infty.$$

We also have that

$$\begin{aligned}
E[M_t^{(2)} \mid \mathcal{F}_s] &= tW_s - \int_0^s W_u du - (t-s)W_s - E\left[\int_s^t (W_u - W_s) du \,\middle|\, \mathcal{F}_s\right] \\
&= sW_s - \int_0^s W_u du \\
&= M_s^{(2)},
\end{aligned}$$

using the conditional Fubini theorem.

Finally, we have that $M^{(3)}$ is adapted and each $M_t^{(3)}$ is integrable since W_t has finite moments (in particular the third moment). We calculate

$$\begin{aligned}
E[M_t^{(3)} \mid \mathcal{F}_s] &= E\big[(W_s + W_t - W_s)^3 \mid \mathcal{F}_s\big] - 3tW_s \\
&= E\Big[W_s^3 + 3W_s^2(W_t - W_s) + 3W_s(W_t - W_s)^2 + (W_t - W_s)^3 \mid \mathcal{F}_s\Big] - 3tW_s \\
&= W_s^3 + 3W_s(t-s) - 3tW_s \\
&= W_s^3 - 3sW_s,
\end{aligned}$$

so $M^{(3)}$ is a martingale, as we wanted.

Exercise 3.4 Let $\rho \in (0,1)$. For a bounded measurable function $f : [0,1] \to \mathbb{R}$, define the moving average function $\mathrm{MA}_\rho f$ by

$$(\mathrm{MA}_\rho f)(t) = \frac{1}{\rho} \int_{t-\rho}^{t} f(u)du,$$

where we set $f(t) = f(0)$ for $t < 0$. Define $\tau(f) = \inf\{t \geq 0 : f(t) \geq (\mathrm{MA}_\rho f)(t) + 1\} \wedge 1$. Show that if X^n is an approximation to a Brownian motion W as in Donsker's theorem, then $\tau(X^n) \to \tau(W)$ in distribution.

Solution 3.4 We need to show that τ is bounded and continuous outside of a nullset under the Wiener measure. Boundedness is clear, since $\tau \leq 1$.

To show continuity, define the set

$$S = \left\{ f \in C([0,1]) : \forall t \in [0,1), \limsup_{\delta \searrow 0} \frac{f(t+\delta) - f(t)}{\delta} = +\infty \right\}.$$

By the law of the iterated logarithm, we know that S has Wiener measure 1. We just need to show that τ is continuous on S.

Indeed, let $f \in S$ and $\varepsilon > 0$. Define

$$\delta = \inf_{s \in [0, \tau(f) - \varepsilon]} \Big(\mathrm{MA}_\rho f(s) + 1 - f(s) \Big) > 0,$$

since f is continuous and by definition of $\tau(f)$. For any $g \in C([0,1])$ such that $\|g\|_\infty < \delta/2$, note that $\|MA_\rho g\|_\infty < \delta/2$ as well, and so it follows that $\tau(f+g) \geq \tau(f) - \varepsilon$.

This already shows that τ is continuous at f if $\tau(f) = 1$, so from now on we assume that $t^* := \tau(f) \in (0,1)$. We want to show that $\tau(f+g) \leq t^* + \varepsilon$ if $\|g\|_\infty$ is small enough. Note that, for $v \in [t^*, t^* + \varepsilon]$,

$$\begin{aligned}
\big|(\mathrm{MA}_\rho f)(v) - (\mathrm{MA}_\rho f)(t^*)\big| &= \left| \frac{1}{\rho} \int_{v-\rho}^{v} f(u)du - \frac{1}{\rho} \int_{t^*-\rho}^{t^*} f(u)du \right| \\
&\leq \frac{1}{\rho} \left(\left| \int_{t^*}^{v} f(u)du \right| + \left| \int_{t^*-\rho}^{v-\rho} f(u)du \right| \right) \\
&\leq M|v - t^*|,
\end{aligned}$$

where $M = \frac{2}{\rho}\|f\|_\infty$. By definition of S, we can find $v \in (t^*, t^* + \varepsilon]$ such that $f(v) \geq f(t^*) + 2M|v - t^*|$. Therefore, we have that

$$f(v) - (\mathrm{MA}_\rho f)(v) - 1 \geq f(t^*) + 2M|v - t^*| - (\mathrm{MA}_\rho f)(t^*) - M|v - t^*| - 1 = M|v - t^*| > 0.$$

It follows that for any $g \in C([0,1])$ such that $\|g\|_\infty < M|v - t^*|/2$, we have that $f(v) + g(v) - (\mathrm{MA}_\rho f)(v) - (\mathrm{MA}_\rho g)(v) - 1 > 0$ and thus $\tau(f+g) \leq \tau(f) + \varepsilon$.

This shows that τ is continuous on S, and the result follows by Donsker's theorem.