Brownian Motion and Stochastic Calculus

Exercise sheet 4

Exercise 4.1 Let W be a Brownian motion on $[0,\infty)$ and $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ constants. The stochastic process $S = (S_t)_{t\geq 0}$ given by

$$S_t = S_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big)$$

is called *geometric Brownian motion.*

(a) Prove that for $\mu \neq \sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = \infty \quad P\text{-a.s.} \qquad \text{or} \qquad \lim_{t\to\infty} S_t = 0 \quad P\text{-a.s.}$$

When do the respective cases arise?

(b) Discuss the behaviour of (S_t) as $t \to \infty$ in the case $\mu = \sigma^2/2$.

(c) Henceforth, suppose that $\mu = 0$. Show that S is a martingale, but not uniformly integrable.

(d) Let τ be a finite stopping time independent of W. Show that $E[S_\tau] = S_0$.

(e) Fix $S_0 = 1$, let $a \in (0,1)$ and let $\tau_a = \inf\{t : S_t \leq a\}$ be its hitting time. Show that $\tau_a < \infty$ almost surely and that $S_{\tau_a} = a < 1$. In particular, $E[S_{\tau_a}] = a < 1 = S_0$.

Solution 4.1

(a) Noting that a.s. $W_t/t \to 0$, we have

- If $(\mu - \sigma^2/2) > 0$, then $\sigma W_t + (\mu - \sigma^2/2)t \to \infty$ a.s., thus $\lim_{t\to\infty} S_t = \infty$.
- If $(\mu - \sigma^2/2) < 0$, then $\sigma W_t + (\mu - \sigma^2/2)t \to -\infty$ a.s., thus $\lim_{t\to\infty} S_t = 0$.

(b) The fact that a.s. $\liminf_{t\to\infty} B_t = -\infty$ and $\limsup_{t\to\infty} B_t = \infty$ implies that when $\mu = \sigma^2/2$, $\liminf_{t\to\infty} S_t = 0$ and $\limsup_{t\to\infty} S_t = \infty$. In particular, (S_t) almost surely does not converge as $t \to \infty$.

(c) Note that if $s \leq t$, we have that $W_t - W_s$ is independent of $\mathcal{F}_s$ and follows the law of a centred normal with variance $t - s$, so that

$$\begin{aligned} E\left[S_t \mid \mathcal{F}_s\right] &= S_0 E\left[\exp(\sigma(W_t - W_s) + \sigma W_s - \sigma^2 t/2) \mid \mathcal{F}_s\right] \\ &= S_0 \exp(\sigma W_s - \sigma^2 s/2) E\left[\exp(\sigma(W_t - W_s) - \sigma^2(t-s)/2\right] = S_s. \end{aligned}$$

Since $S \geq 0$, the same calculation with $s = 0$ shows that S_t is integrable for all $t \in [0,\infty)$. Thus, (S_t) is a martingale that converges to 0 a.s. due to (a). By contradiction, suppose that it is uniformly integrable. We should have then $S_0 = E\left[S_\infty\right] = 0$, which does not hold.

(d) Since τ is independent of W and hence also of S, we can condition on τ to find that

$$E[S_\tau] = E\big[E[S_\tau \mid \tau]\big] = E\big[E[S_t]|_{t=\tau}\big] = S_0$$

because S is a martingale.

(e) Since $\lim_{t\to\infty} S_t = 0$ a.s., it follows that $P[\exists\, t \geq 0 : S_t \leq a] = 1$, and therefore $P[\tau_a < \infty] = 1$. As $S_0 = 1$, we have that $P[\tau_a > 0] = 1$. Note S is P-a.s. continuous; thus for some $\tilde{\Omega} \subseteq \Omega$ with $P[\tilde{\Omega}] = 1$, we have for all $\omega \in \tilde{\Omega}$ that $S_{\tau_a(\omega)}(\omega) = \lim_{t\nearrow\tau_a(\omega)} S_t(\omega) \geq a$, since $S_t(\omega) > a$ for $0 \leq t < \tau_a(\omega)$, and $S_{\tau_a(\omega)}(\omega) = \lim_{t\searrow\tau_a(\omega)} S_t(\omega) \leq a$, since for any $\varepsilon > 0$, there exists $t \in [\tau_a(\omega), \tau_a(\omega) + \varepsilon]$ such that $S_t(\omega) \leq a$. Therefore we must have that $S_{\tau_a(\omega)}(\omega) = a$ for $\omega \in \tilde{\Omega}$, i.e. $S_{\tau_a} = a$ P-a.s.

Exercise 4.2 Consider two stopping times σ, τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. The goal of this exercise, together with exercise **3.1**, is to show that

$$E[E[\,\cdot\,|\mathcal{F}_\sigma]|\mathcal{F}_\tau] = E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}] = E[E[\,\cdot\,|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \quad P\text{-a.s.}, \tag{$\star$}$$

i.e., the operators $E[\,\cdot\,|\mathcal{F}_\tau]$ and $E[\,\cdot\,|\mathcal{F}_\sigma]$ commute and their composition equals $E[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.
Remark: For arbitrary sub-σ-algebras $\mathcal{G}, \mathcal{G}' \subseteq \mathcal{F}$, the conditional expectations $E[E[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $E[E[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $E[\,\cdot\,|\mathcal{G}\cap\mathcal{G}']$ do **not** coincide in general.

(a) Show that if Y is $\mathcal{F}_\sigma$-measurable, then $Y\mathbb{1}_{\{\sigma\le\tau\}}$ and $Y\mathbb{1}_{\{\sigma<\tau\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(b) Show that $E[Y|\mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable if Y is an integrable $\mathcal{F}_\sigma$-measurable random variable. Conclude ($\star$).

(c) Let $M = (M_t)_{t\ge0}$ be a martingale with all trajectories right-continuous. Show that the stopped process $M^\tau = (M_{\tau\wedge t})_{t\ge0}$ is again a martingale.
Hint: Use ($\star$) and the stopping theorem.

Solution 4.2

(a) Since $\{\sigma\le\tau\}, \{\sigma<\tau\} \in \mathcal{F}_\sigma$ by exercise **3.1(a)**, we have that $Y\mathbb{1}_{\{\sigma\le\tau\}}, Y\mathbb{1}_{\{\sigma<\tau\}}$ are both $\mathcal{F}_\sigma$-measurable. Now let us prove that they are $\mathcal{F}_\tau$-measurable. This holds if Y takes finitely many values. Indeed, let $Y^n = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$ for some $A_1, \dots, A_n \in \mathcal{F}_\sigma$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then $Y^n\mathbb{1}_{\{\sigma\le\tau\}}$ is $\mathcal{F}_\tau$-measurable if $A_i \cap \mathbb{1}_{\{\sigma\le\tau\}}$ is $\mathcal{F}_\tau$-measurable for each i, which holds by exercise **3.1(b)**. The argument for $Y^n\mathbb{1}_{\{\sigma<\tau\}}$ is analogous.

For general Y, we can construct simple random variables Y^n of the above form such that $Y^n(\omega) \to Y(\omega)$ for all $\omega\in\Omega$, and thus $Y^n\mathbb{1}_{\sigma\le\tau} \to Y\mathbb{1}_{\{\sigma\le\tau\}}$, which is therefore $\mathcal{F}_\tau$-measurable, and likewise for $Y\mathbb{1}_{\{\sigma<\tau\}}$. By exercise **3.1(a)**, we conclude that $Y\mathbb{1}_{\{\sigma\le\tau\}}$ and $Y\mathbb{1}_{\{\sigma<\tau\}}$ are $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

(b) We note that

$$E\left[Y \mid \mathcal{F}_\tau\right] = E\left[Y\mathbb{1}_{\{\tau<\sigma\}} \mid \mathcal{F}_\tau\right] + E\left[Y\mathbb{1}_{\{\sigma\le\tau\}} \mid \mathcal{F}_\tau\right] = E\left[Y \mid \mathcal{F}_\tau\right]\mathbb{1}_{\tau<\sigma} + Y\mathbb{1}_{\sigma\le\tau},$$

where each term is $\mathcal{F}_{\sigma\wedge\tau}$-measurable by (a) so that $E\left[Y \mid \mathcal{F}_\tau\right]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable.

To show ($\star$) it is enough to note that if Z is integrable, then $E\left[Z \mid \mathcal{F}_\sigma\right]$ is $\mathcal{F}_\sigma$-measurable and $E\left[E\left[Z \mid \mathcal{F}_\sigma\right] \mid \mathcal{F}_\tau\right]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable. Therefore

$$E\big[E\left[Z \mid \mathcal{F}_\sigma\right] \mid \mathcal{F}_\tau\big] = E\Big[E\big[E\left[Z \mid \mathcal{F}_\sigma\right] \mid \mathcal{F}_\tau\big] \mid \mathcal{F}_{\sigma\wedge\tau}\Big] = E\big[E\left[Z \mid \mathcal{F}_\sigma\right] \mid \mathcal{F}_{\sigma\wedge\tau}\big] = E\left[Z \mid \mathcal{F}_{\sigma\wedge\tau}\right],$$

by the tower property. The other direction follows by symmetry.

(c) Take $s\le t$ and note that $\tau\wedge s \le \tau\wedge t$ are bounded stopping times. By the stopping theorem,

$$E\left[M_{\tau\wedge t} \mid \mathcal{F}_s\right] = E\left[E\left[M_t \mid \mathcal{F}_{\tau\wedge t}\right] \mid \mathcal{F}_s\right] = E\left[M_t \mid \mathcal{F}_{\tau\wedge s}\right] = M_{\tau\wedge s}$$

where in the second equality we used ($\star$).

Exercise 4.3 Let $(S, \mathcal{S})$ be a measurable space, let $Y = (Y_t)_{t\geq 0}$ be the canonical process on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$, i.e., $Y_t(y) = y(t)$ for $y \in S^{[0,\infty)}$, $t \geq 0$, and let $(K_t)_{t\geq 0}$ be a transition semigroup on $(S, \mathcal{S})$. Moreover, for each $x \in S$, assume that there exists a unique probability measure $\mathbb{P}_x$ on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ under which Y is a Markov process with transition semigroup $(K_t)_{t\geq 0}$ and initial distribution $\nu = \delta_{\{x\}}$.

Suppose $Z \geq 0$ is an $\mathcal{S}^{[0,\infty)}$-measurable random variable on $S^{[0,\infty)}$. Use the monotone class theorem to prove that the map $x \mapsto \mathbb{E}_x[Z], x \in S$, is $\mathcal{S}$-measurable.

Solution 4.3 Let $\mathcal{H}$ denote the real vector space of all bounded, $\mathcal{S}^{[0,\infty)}$-measurable functions $Z : S^{[0,\infty)} \to \mathbb{R}$ such that the map $x \mapsto \mathbb{E}_x[Z]$, $x \in S$, is $\mathcal{S}$-measurable. Since pointwise limits of measurable functions are measurable, $\mathcal{H}$ is closed under monotone bounded convergence. The family

$$\mathcal{M} = \left\{ \prod_{k=0}^{n} f_k(Y_{t_k}) : n \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_n, f_k : S \to \mathbb{R} \ \mathcal{S}\text{-measurable and bounded} \right\}$$

is closed under multiplication and $\sigma(\mathcal{M}) = \mathcal{S}^{[0,\infty)}$. It remains to show that $\mathcal{M} \subseteq \mathcal{H}$ (note that $1 \in \mathcal{M}$). Indeed, for an element $Z = \prod_{k=0}^{n} f_k(Y_{t_k})$ in $\mathcal{M}$, we have for all $x \in S$ that

$$\begin{aligned} \mathbb{E}_x[Z] &= \int_S \delta_{\{x\}}(dx_0) f_0(x_0) \int_S K_{t_1 - t_0}(x_0, dx_1) f_1(x_1) \cdots \int_S K_{t_n - t_{n-1}}(x_{n-1}, dx_n) f_n(x_n) \\ &= f_0(x) \int_S K_{t_1 - t_0}(x, dx_1) f_1(x_1) \cdots \int_S K_{t_n - t_{n-1}}(x_{n-1}, dx_n) f_n(x_n). \end{aligned} \quad (1)$$

Using measure-theoretic induction, it is easy to see that $x \mapsto \int_S f(y) K(x, dy)$, $x \in S$, is $\mathcal{S}$-measurable for any bounded, $\mathcal{S}$-measurable function $f : S \to \mathbb{R}$ and any stochastic kernel K on $(S, \mathcal{S})$ (in fact, a more general result follows from the proof of Fubini's theorem for measures of the form $\nu \otimes K$, see for instance lecture notes "Wahrscheinlichkeitstheorie" (Föllmer/Schweizer), proof of Theorem II.1.4). Using this fact inductively in (1), we conclude that $x \mapsto \mathbb{E}_x[Z]$ is $\mathcal{S}$-measurable for $Z \in \mathcal{M}$.

The monotone class theorem implies that $\mathcal{H}$ contains all bounded, $\mathcal{S}^{[0,\infty)}$-measurable Z. For a general $\mathcal{S}^{[0,\infty)}$-measurable $Z \geq 0$, we have for each $x \in S$ that $\mathbb{E}_x[Z] = \lim_{n\to\infty} \mathbb{E}_x[Z \wedge n]$ by monotone convergence. Thus, as a pointwise limit of $\mathcal{S}$-measurable functions, $x \mapsto \mathbb{E}_x[Z]$ is $\mathcal{S}$-measurable.

Exercise 4.4 Part (a) of this exercise is optional, but the results are needed in (b) and (c).

(a) Let $L \in \mathbb{N}$ and consider a matrix $Q \in \mathbb{R}^{L \times L}$. For $t \in [0, +\infty)$, define $\exp(tQ) := \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!}$.

1. Show that $\exp(tQ)$ is well-defined for any $t \in [0, +\infty)$ and $\exp(0Q) = \mathbb{I}$, the identity matrix.
2. Show that $Q^n \exp(tQ) = \exp(tQ) Q^n$ and that $\exp(sQ)\exp(tQ) = \exp(tQ)\exp(sQ) = \exp((t+s)Q)$ for any $n \in \mathbb{N}$ and $s, t \geq 0$.
3. Show that
$$\lim_{h \searrow 0} \frac{\exp((t+h)Q) - \exp(tQ)}{h} = Q \exp(tQ).$$
4. Show that
$$\lim_{M \to \infty} \left(1 + \frac{tQ}{M}\right)^M = \exp(tQ).$$

(b) Consider $S = \{x_1, \dots, x_L\} \subseteq \mathbb{R}$. Define the operators $(K_t)_{t \geq 0}$ by
$$K_t(x_i, A) = \sum_{x_j \in A} (\exp(tQ))_{ij}, \qquad \text{for } A \subseteq S.$$
Show that $K_{s+t}(x_i, \{x_\ell\}) = \sum_{j=1}^{L} K_s(x_i, \{x_j\}) K_t(x_j, \{x_\ell\})$ for $s, t \geq 0$.

(c) Suppose that there exists a Markov process X taking values in S such that
$$\mathbb{P}_{x_i}[X_t = x_j] = K_t(x_i, \{x_j\}) = (\exp(tQ))_{ij}. \tag{2}$$
Noting the fact that for all $t \geq 0$ and $x_i \in S$, the map $A \mapsto K_t(x_i, A)$ must then be a probability measure, what does this imply about Q?

Solution 4.4

(a)
1. Let $\|\cdot\|$ be the operator norm for a matrix, defined as
$$\|A\| = \sup_{v \neq 0} \frac{|Av|}{|v|}.$$
The set of matrices $\mathbb{R}^{L \times L}$ is a Banach space with respect to this norm. For any $n \geq 0$, we have the inequality $\|A^n\| \leq \|A\|^n$, and therefore
$$\sum_{n=0}^{\infty} \left\| \frac{t^n Q^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{t^n \|Q\|^n}{n!} = \exp(t\|Q\|) < \infty.$$
Therefore, $\exp(tQ)$ is well-defined as an absolutely convergent series in $\mathbb{R}^{L \times L}$.
2. We have that
$$Q^n \exp(tQ) = Q^n \sum_{m=0}^{\infty} \frac{t^m Q^m}{m!} = \sum_{m=0}^{\infty} \frac{t^m Q^{m+n}}{m!} = \exp(tQ) Q^n$$
and
$$\exp(sQ)\exp(tQ) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{s^m t^n Q^{m+n}}{m! n!} = \sum_{l=0}^{\infty} \frac{Q^l}{l!} \sum_{m+n=l} \binom{l}{m} s^m t^n = \sum_{l=0}^{\infty} \frac{(s+t)^l Q^l}{l!} = \exp\big((s+t)Q\big).$$

3. We have that

$$\exp((t+h)Q) = \exp(tQ)\exp(hQ) = \exp(tQ)\left(\mathbb{I} + hQ + O(h^2)\right)$$

and therefore
$$\frac{d}{dt}\exp(tQ) = \exp(tQ)Q = Q\exp(tQ).$$

4. For each $k \in \mathbb{N}$, we have that

$$\lim_{M\to\infty}\binom{M}{k}\frac{(tQ)^k}{M^k} = \lim_{M\to\infty}\frac{M(M-1)\cdots(M-k+1)}{M^k}\frac{(tQ)^k}{k!} = \frac{(tQ)^k}{k!}.$$

Moreover, we have the uniform bound $\left\|\binom{M}{k}\frac{(tQ)^k}{M^k}\right\| \le \frac{(t\|Q\|)^k}{k!}$, which is summable. Therefore, by the dominated convergence theorem (with respect to the counting measure),

$$\lim_{M\to\infty}\left(\mathbb{I}+\frac{tQ}{M}\right)^M = \lim_{M\to\infty}\sum_{k=0}^{M}\binom{M}{k}\frac{(tQ)^k}{M^k} = \sum_{k=0}^{\infty}\frac{(tQ)^k}{k!} = \exp(tQ).$$

(b) We calculate

$$\begin{aligned}\sum_{j=1}^{L}K_s(x_i,\{x_j\})K_t(x_j,\{x_\ell\}) &= \sum_{j=1}^{L}(\exp(sQ))_{ij}\exp(tQ)_{j\ell}\\ &= \big(\exp(sQ)\exp(tQ)\big)_{i\ell}\\ &= \exp\big((s+t)Q\big)_{i\ell}\\ &= K_{s+t}(x_i,\{x_\ell\}).\end{aligned}$$

(c) Suppose that such a Markov process exists. Note that for each $i,j\in\{1,\dots,L\}$, we have that $(\exp(0Q))_{ij} = \mathbb{I}_{ij} = \mathbb{1}_{\{i=j\}}$, by (a1). From (a3), we find that for $i\neq j$,

$$\lim_{h\searrow 0}\frac{\mathbb{P}_{x_i}[X_h = x_j]}{h} = \lim_{h\searrow 0}\frac{\exp(hQ)_{ij}-\exp(0Q)_{ij}}{h} = Q_{ij}$$

and
$$\lim_{h\searrow 0}\frac{\mathbb{P}_{x_i}(X_h = x_j)-1}{h} = \lim_{h\searrow 0}\frac{\exp(hQ)_{ii}-\exp(0Q)_{ii}}{h} = Q_{ii}.$$

for $i\in\{1,\dots,L\}$.

Since X exists, the operator K_t must be a transition kernel, and therefore $A\mapsto K_t(x_i,A)$ must be a probability measure for each $t\ge 0$ and $x_i\in S$. In particular, note that

$$K_t(x_i,S) = 1 = \sum_{j=1}^{L}(\exp(tQ))_{ij}$$

and therefore $\sum_{j=1}^{L}(\exp(tQ))_{ij} = 1$ must hold for all $t\ge 0$. Taking the derivative at $t=0$, we find that $\sum_{j=1}^{L}Q_{ij} = 0$ for each i.

Moreover, we must have that $K_t(x_i,\{x_j\}) = (\exp(tQ))_{ij}\ge 0$ for all i,j. Since $(\exp(tQ))_{ij} = 0$ for $t=0$ and $i\neq j$, we can take the derivative to find that $Q_{ij}\ge 0$ for all $i\neq j$.

We conclude that the conditions $Q_{ij}\ge 0$ for $i\neq j$ and $Q_{ii} = -\sum_{i\neq j}Q_{ij}$ are necessary. One can show that they are also sufficient.