Brownian Motion and Stochastic Calculus

Exercise sheet 5

Exercise 5.1

(a) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space. Assume that $\Omega = \{\omega_1, \dots, \omega_k\}$ is finite and that $\mathcal{F} = 2^\Omega$.

Show that the $\mathbb{R}^k$-valued process

$$X_t = \big(P[\{\omega_1\} \mid \mathcal{F}_t], \dots, P[\{\omega_k\} \mid \mathcal{F}_t]\big)^\top$$

is a Markov process.

(b) Let W be a Brownian motion. Which of the following processes X are Markov? Write down the corresponding transition kernels in those cases.

1. $X_t = |W_t|$ (reflected Brownian motion).
2. $X_t = \int_0^t W_u du$ (integrated Brownian motion).
3. $X_t = W_{\tau_a \wedge t}$, where $\tau_a = \inf\{t \geq 0 : W_t \geq a\}$ is the hitting time of $a > 0$.
4. $X_t = W_t^\tau$ for a random time $\tau \sim \mathrm{Exp}(1)$ independent of W.
5. $X_t = t - t \wedge \tau$, where $\tau \sim \mathrm{Exp}(1)$ is a random time.

Solution 5.1

(a) Let $s \leq t$ and let g be a bounded measurable function. For $\omega \in \Omega$, we have that

$$\left(E[g(X_t) \mid \mathcal{F}_s]\right)(\omega) = \sum_{i=1}^k g(X_t(\omega_i)) \left(P[\{\omega_i\} \mid \mathcal{F}_s]\right)(\omega) = \sum_{i=1}^k g(X_t(\omega_i)) X_s^i(\omega).$$

Note that the $g(X_t(\omega_i))$ are constants which do not depend on ω. Therefore, the conditional expectation is a (linear) function of X_s, so the process is Markov.

(b) 1. This is a Markov process. Let $(\mathcal{F}_t^W), (\mathcal{F}_t^{|W|})$ be the filtrations generated by $W, |W|$ respectively. For Borel $A \subseteq [0, \infty)$, $t \geq 0$ and $h > 0$, we have that

$$\begin{aligned}
P\left[|W_{t+h}| \in A \,\middle|\, \mathcal{F}_t^W\right] &= P\left[W_{t+h} \in A \,\middle|\, \mathcal{F}_t^W\right] + P\left[-W_{t+h} \in A \,\middle|\, \mathcal{F}_t^W\right] \\
&= \int_A \frac{1}{\sqrt{2\pi h}} \left(e^{-\frac{(y-W_t)^2}{2h}} + e^{-\frac{(y+W_t)^2}{2h}}\right) dy \\
&= \int_A \frac{1}{\sqrt{2\pi h}} \left(e^{-\frac{(y-|W_t|)^2}{2h}} + e^{-\frac{(y+|W_t|)^2}{2h}}\right) dy.
\end{aligned}$$

By the tower law and since this is $\mathcal{F}_t^{|W|}$-measurable, this is also

$$P\left[|W_{t+h}| \in A \,\middle|\, \mathcal{F}_t^{|W|}\right] = K_h(|W_t|, A) = P\left[|W_{t+h}| \in A \mid \sigma(|W_t|)\right],$$

so X is Markov.

2. This is not a Markov process. Let $(\mathcal{F}_t^X)$ be the filtration generated by X. For Borel $A \subseteq \mathbb{R}$,
$$P\left[X_t \in A \mid \mathcal{F}_s^W\right] = P\left[X_s + (t-s)W_s + \int_s^t (W_r - W_s)dr \in A \,\middle|\, \mathcal{F}_s^W\right]$$
$$= f_{t-s}\big(X_s + (t-s)W_s, A\big),$$
where $f_t(x, A) = P[x + \int_0^t W_r dr \in A]$, using the Markov property of W. We also note that $\mathcal{F}_t^W = \mathcal{F}_t^X$, where "$\supseteq$" is immediate and "$\subseteq$" follows by $W_s = \lim_{\varepsilon \searrow 0} \frac{X_s - X_{s-\varepsilon}}{\varepsilon}$. Therefore,
$$P[X_t \in A \mid \mathcal{F}_s^X] = P[X_t \in A \mid \mathcal{F}_s^W] = f_{t-s}\big(X_s + (t-s)W_s, A\big).$$
But $x \mapsto f_t(x, A)$ is injective (strictly increasing) for $A = [0, \infty)$ and W_s is not $\sigma(X_s)$-measurable (this follows from exercise **4.3**), so X is not Markov.

3. This is a Markov process. Let $(\mathcal{F}_t^X)$ be the filtration generated by X. For Borel $A \subseteq \mathbb{R}$, define $f_t^a(w, A) = P[w + W_{t \wedge \tau_{a-w}} \in A]$ for $t \geq 0$ and $0 \leq w \leq a$. Note that $\{\tau_a < t\} \in \mathcal{F}_t^W$ for all $t > 0$, and moreover
$$\{X_t = a\} = \{\tau_a \leq t\} = \{\tau_a < t\} \cup \{\tau_a = a\}.$$
Since $f_h^a(a, A) = \delta_a(A)$, we find that
$$P[X_{t+h} \in A \mid \mathcal{F}_t^W] = \mathbb{1}_{\{\tau_a < t\}}\delta_a(A) + f_h^a(W_t, A)\mathbb{1}_{\{\tau_a \geq t\}}$$
$$= \mathbb{1}_{\{X_t = a\}}\delta_a(A) + f_h^a(X_t, A)\mathbb{1}_{\{X_t < a\}},$$
where the first line is justified by the Markov property of W, and the second one follows since $\{\tau_a \leq t\} = \{X_t = a\}$. Since this is $\mathcal{F}_t^X$-measurable and $\mathcal{F}_t^X \subseteq \mathcal{F}_t^W$, we have that
$$P[X_{t+h} \in A \mid \mathcal{F}_t^X] = P[X_{t+h} \in A \mid \mathcal{F}_t^W] = \mathbb{1}_{\{X_t = a\}}\delta_a(A) + f_h^a(X_t, A)\mathbb{1}_{\{X_t < a\}},$$
which is $\sigma(X_t)$-measurable, so X is Markov.
Remark: One can show that
$$f_t^a(w, (-\infty, y]) = \Phi\left(\frac{2a - y - w}{\sqrt{t}}\right) - \Phi\left(\frac{-y + w}{\sqrt{t}}\right),$$
for Φ the distribution function of a standard Gaussian and any $y < a$, while $f_t^a(w, \{a\}) = 2\Phi\big(\frac{-a+w}{\sqrt{t}}\big)$.

4. This is not a Markov process. Note that $\{\tau < t\} \in \mathcal{F}_t^X$ for each $t > 0$, since
$$\{\tau < t\} = \bigcup_{q \in (0,t) \cap \mathbb{Q}} \bigcap_{r \in (q,t) \cap \mathbb{Q}} \{X_r = X_q\} \in \mathcal{F}_t^X$$
as X stays constant after τ. Therefore,
$$P[X_t \in A \mid \mathcal{F}_s^X]\mathbb{1}_{\{\tau < s\}} = \delta_{X_s}(A)\mathbb{1}_{\{\tau < s\}}.$$
However, $\{\tau < s\} \notin \sigma(X_s)$; therefore X is not Markov.

5. This is a Markov process. Note that the filtration is
$$\mathcal{F}_t^X = \sigma(\tau \wedge t) = \sigma(X_t),$$
since $\sigma(X_s) = \sigma(\tau \wedge s) = \sigma((\tau \wedge t) \wedge s) \subseteq \sigma(X_t)$ for $s \leq t$. Therefore, it follows immediately that X is Markov. Note that on $\{X_t > 0\}$, it holds that $\tau < t$ and therefore $X_{t+h} = X_t + h$ a.s. On the other hand, on $\{X_t = 0\} = \{\tau \geq t\}$, we have that $(\tau \mid \{\tau \geq t\}) \sim t + \mathrm{Exp}(1)$ by the memoryless property of the exponential distribution, and therefore $(X_{t+h} \mid X_t) \sim 0 \vee (h - \mathrm{Exp}(1))$. This allows us to compute the kernel
$$K_h(x, A) = \mathbb{1}_{\{x>0\}}\delta_{x+h}(A) + \mathbb{1}_{\{x=0\}}\left(e^{-h}\delta_{\{0\}}(A) + \int_0^h e^{-s}\mathbb{1}_{\{s \in A\}}ds\right).$$

Exercise 5.2 Let $W = (W_t)_{t\geq 0}$ be a Brownian motion and define the *integrated Brownian motion* $Y = (Y_t)_{t\geq 0}$ by $Y_t = \int_0^t W_s ds$. Moreover, let $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\geq 0}$ be the raw filtration generated by W.

(a) For each $h \geq 0$, show that the pair (W_h, Y_h) has a two-dimensional normal distribution with mean zero and covariance matrix given by

$$\begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}.$$

Hint: You may want to apply Donsker's theorem by constructing a continuous mapping $F : C([0,\infty)) \to \mathbb{R}^2$ such that $F((W_t)_{t\geq 0}) = (W_h, Y_h)$. You may also use a result on weak convergence of Gaussian random variables.

(b) Show that the pair (W, Y) is a (homogeneous) Markov process with state space $\mathbb{R}^2$, filtration $\mathbb{F}^W = \mathbb{F}^{W,Y}$ and transition semigroup $(K_h)_{h\geq 0}$ given by

$$K_h\big((w,y),\cdot\big) = \mathcal{N}\left(\begin{pmatrix} w \\ y+hw \end{pmatrix}, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}\right), \quad h \geq 0.$$

Solution 5.2

(a) We assume for convenience that $h = 1$, since otherwise we can obtain the result by rescaling: note that $(W_t)_{t\geq 0} \overset{d}{=} (\lambda^{-1} W_{\lambda^2 t})_{t\geq 0}$ for any $\lambda > 0$, so that

$$\begin{aligned}(W_h, Y_h) = \left(W_h, \int_0^h W_s ds\right) &\overset{d}{=} \left(\sqrt{h}W_1, \sqrt{h}\int_0^h W_{s/h} ds\right) \\ &= \left(\sqrt{h}W_1, \sqrt{h^3}\int_0^1 W_u du\right) = \left(\sqrt{h}W_1, \sqrt{h^3}Y_1\right),\end{aligned}$$

from which we can compute the distribution of (W_t, Y_t) given that of (W_1, Y_1).

Consider the mapping $F : C([0,\infty)) \to \mathbb{R}^2$ given by

$$F\big((X_t)_{t\geq 0}\big) = \left(X_1, \int_0^1 X_s\right).$$

This is clearly continuous on $C([0,\infty))$. We can construct an approximation (X^n) to W in two ways:

1. As the Donsker approximation to Brownian motion, with jumps Z_j distributed according to $\mathcal{N}(0,1)$, in which case $X^n \Rightarrow W$. Then,

$$F\big((X_t^n)_{t\geq 0}\big) = (X_1^n, Y_1^n) = \left(\frac{1}{\sqrt{n}}\sum_{j=1}^n Z_j, \frac{1}{n\sqrt{n}}\sum_{j=1}^n \left(j - \frac{1}{2}\right) Z_j\right) \Rightarrow F(W) = (W_1, Y_1).$$

2. As the linear interpolation $X_t^n = n(k(t) + n^{-1} - t)W_{k(t)} + n(t - k(t))W_{k(t)+n^{-1}}$ for $k(t) = \lfloor nt \rfloor / n$, in which case we have pointwise convergence $X^n \to W$. This of course implies $X^n \Rightarrow W$.

Since the two approximations (X^n) have the same law and $(X^n) \Rightarrow W$ in both cases, either will work for the rest of the proof.

Since the increments are Gaussian, we can compute the law

$$(X_1^n, Y_1^n) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1, & \frac{1}{n^2}\ \sum_{j=1}^n \left(j - \frac{1}{2}\right), \\ \frac{1}{n^2}\ \sum_{j=1}^n \left(j - \frac{1}{2}\right), & \frac{1}{n^3}\ \sum_{j=1}^n \left(j - \frac{1}{2}\right)^2 \end{pmatrix}\right).$$

Taking $n \to \infty$, the limiting terms in the matrix can be calculated as Riemann sums:

$$\frac{1}{n^2}\ \sum_{j=1}^{n} \left(j - \frac{1}{2}\right) = \sum_{j/n=1}^{1} \left(\frac{j}{n} - \frac{1}{2n}\right)\frac{1}{n} \to \int_0^1 x dx = \frac{1}{2},$$

and likewise

$$\frac{1}{n^3}\ \sum_{j=1}^{n} \left(j - \frac{1}{2}\right)^2 \to \int_0^1 x^2 dx = \frac{1}{3},$$

noting that in both cases the term $(\frac{1}{2})$ becomes negligible. We recall the theorem about weak convergence of Gaussian random variables: if $Z^n \sim \mathcal{N}(\mu_n, \Sigma_n)$ with $\mu_n \to \mu \in \mathbb{R}^k$ and $\Sigma_n \to \Sigma \in \mathbb{R}^{k\times k}$, then $Z^n \Rightarrow Z$, where $Z \sim \mathcal{N}(\mu, \Sigma)$. Applying this result, we get that

$$(X_1, Y_1) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1, & \frac{1}{2} \\ \frac{1}{2}, & \frac{1}{3} \end{pmatrix}\right),$$

as we wanted, which then gives the distribution of (X_t, Y_t).

(b) Let $(\mathcal{F}_t^W)_{t\geq 0}$ denote the (raw) filtration generated by W, $f : \mathbb{R}^2 \to [0, \infty)$ a bounded Borel function, and $t \geq 0$, $h > 0$. By construction, Y is $\mathbb{F}^W$-adapted. Moreover, writing

$$\begin{aligned} W_{t+h} &= W_t + (W_{t+h} - W_t), \\ Y_{t+h} &= Y_t + \int_t^{t+h} W_u\, du = Y_t + hW_t + \int_t^{t+h} (W_u - W_t)\, du \end{aligned}$$

and using the fact that W is Markov, we obtain

$$E[f(W_{t+h}, Y_{t+h})|\mathcal{F}_t^W] = g_h(W_t, Y_t),$$

where $g_h(x, y) = E\left[f\left(w + W_h, y + hw + \int_0^h W_u\, du\right)\right]$. Thus, by part (a),

$$E[f(W_{t+h}, Y_{t+h})|\mathcal{F}_t^W] = \int_{\mathbb{R}^2} f(x) K_h\big((W_t, Y_t), dx\big),$$

i.e., (W, Y) is a Markov process with transition semigroup $(K_h)_{h\geq 0}$ given by

$$K_h((w, y), \cdot) = \mathcal{N}\left(\begin{pmatrix} w \\ y + hw \end{pmatrix}, \begin{pmatrix} h & h^2/2 \\ h^2/2 & h^3/3 \end{pmatrix}\right), \quad h \geq 0.$$

Exercise 5.3

(a) Recall the canonical space $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ of all real-valued functions equipped with the σ-algebra generated by all projections. Let $\lambda > 0$, $x \in \mathbb{R}$ and construct on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ a probability measure P such that the canonical process $(Y_t)_{t\geq 0}$ has independent increments, satisfies $P[Y_0 = x] = 1$ and $Y_t - Y_s \sim \text{Poi}(\lambda(t-s))$.

Hint: Use the Kolmogorov consistency theorem.

(b) A Poisson process is a process $(N_t)_{t\geq 0}$ such that all trajectories are RCLL and piecewise constant, all jumps are of size $+1$, and the increments $N_t - N_s \sim \text{Poi}(\lambda(t-s))$ are independent. Show that the process $(Y_t)_{t\geq 0}$ defined in (a) admits a version which is a Poisson process.

(c) Let $(N_t)_{t\geq 0}$ be a Poisson process. For $n \in \mathbb{N}$, find the distribution of the random variables

$$\tau_n = \inf\{t \geq 0 : N_t - N_0 = n\}.$$

(d) Show that N is a Markov process.

Solution 5.3

(a) For $0 \leq t_1 < \cdots < t_m$, we set $I = \{t_1, \ldots, t_m\}$ and construct a measure $\nu^{(I)}$ on $\mathbb{R}^{|I|} = \mathbb{R}^m$ by

$$\nu^{(I)}\big[\{x + k_1, \ldots, x + k_m\}\big] = \prod_{j=1}^{m} e^{-\lambda(t_j - t_{j-1})} \frac{(\lambda(t_j - t_{j-1}))^{k_j - k_{j-1}}}{(k_j - k_{j-1})!}$$

for $k_1, \ldots, k_m \in \mathbb{N}$ with $k_1 \leq \cdots \leq k_m$ and letting $t_0 = 0$ and $k_0 = 0$. We assign no mass outside of points of this form.

We show that $\{\nu^{(I)} : I \subseteq [0,\infty) \text{ finite}\}$ is a consistent family of measures. By taking a union of the time indices, we can w.l.o.g. restrict ourselves to checking that $\nu^{(I)}$ and $\nu^{(J)}$ are consistent for $I = \{t_1, \ldots, t_m\}$ and $J = \{t_{\ell_1}, \ldots, t_{\ell_n}\} \subseteq I$, where $n \leq m$, $0 \leq t_1 < \cdots < t_m$ and $1 \leq \ell_1 < \cdots < \ell_n \leq m$. For convenience, set $\ell_0 = 0$ so that $t_{\ell_0} = t_0 = 0$. For each $i \in \{1, \ldots, m\}$, we have the following identity by the multinomial formula:

$$\begin{aligned}
&\frac{\big(\lambda(\sum_{j=\ell_{i-1}+1}^{\ell_i}(t_j - t_{j-1}))\big)^{k_{\ell_i} - k_{\ell_{i-1}}}}{(k_{\ell_i} - k_{\ell_{i-1}})!} \\
&= \frac{1}{(k_{\ell_i} - k_{\ell_{i-1}})!} \sum_{\sum a_j = k_{\ell_i} - k_{\ell_{i-1}}} \binom{k_{\ell_i} - k_{\ell_{i-1}}}{a_{\ell_{i-1}+1}, \ldots, a_{\ell_i}} \prod_{j=\ell_{i-1}+1}^{\ell_i} (\lambda(t_j - t_{j-1}))^{a_j} \\
&= \sum_{\sum a_j = k_{\ell_i} - k_{\ell_{i-1}}} \prod_{j=\ell_{i-1}+1}^{\ell_i} \frac{(\lambda(t_j - t_{j-1}))^{a_j}}{a_j!} \\
&= \sum_{\substack{\tilde{k}_{\ell_{i-1}} \leq \cdots \leq \tilde{k}_{\ell_i} \\ \tilde{k}_{\ell_{i-1}} = k_{\ell_{i-1}} \\ \tilde{k}_{\ell_i} = k_{\ell_i}}} \prod_{j=\ell_{i-1}+1}^{\ell_i} \frac{(\lambda(t_j - t_{j-1}))^{\tilde{k}_j - \tilde{k}_{k-1}}}{(\tilde{k}_j - \tilde{k}_{j-1})!},
\end{aligned}$$

where $\sum a_j = \sum_{j=\ell_{i-1}+1}^{\ell_i} a_j$ and setting $\tilde{k}_j = \tilde{k}_{j-1} + a_j$.

Using this identity we find

$$
\begin{aligned}
&\nu^{(J)}\left[\{x+k_{\ell_1},\dots,x+k_{\ell_n}\}\right]\\
&= \prod_{i=1}^{n} e^{-\lambda(t_{\ell_i}-t_{\ell_{i-1}})}\frac{(\lambda(t_{\ell_i}-t_{\ell_{i-1}}))^{k_{\ell_i}-k_{\ell_{i-1}}}}{(k_{\ell_i}-k_{\ell_{i-1}})!}\\
&= \prod_{i=1}^{n} \exp\left(-\lambda\sum_{j=\ell_{i-1}+1}^{\ell_i}(t_j-t_{j-1})\right)\frac{\left(\lambda(\sum_{j=\ell_{i-1}+1}^{\ell_i}(t_j-t_{j-1}))\right)^{k_{\ell_i}-k_{\ell_{i-1}}}}{(k_{\ell_i}-k_{\ell_{i-1}})!}\\
&= \prod_{i=1}^{n}\sum_{\substack{\tilde k_{\ell_{i-1}}\le\cdots\le\tilde k_{\ell_i}\\ \tilde k_{\ell_{i-1}}=k_{\ell_{i-1}}\\ \tilde k_{\ell_i}=k_{\ell_i}}}\prod_{j=\ell_{i-1}+1}^{\ell_i} e^{-\lambda(t_j-t_{j-1})}\frac{(\lambda\,(t_j-t_{j-1}))^{\tilde k_j-\tilde k_{j-1}}}{(\tilde k_j-\tilde k_{j-1})!}\\
&= \sum_{\substack{\tilde k_0\le\cdots\le\tilde k_m\\ \forall i:\,\tilde k_{\ell_i}=k_{\ell_i}}}\prod_{j=1}^{m} e^{-\lambda(t_j-t_{j-1})}\frac{(\lambda\,(t_j-t_{j-1}))^{\tilde k_j-\tilde k_{j-1}}}{(\tilde k_j-\tilde k_{j-1})!}\\
&= \nu^{(I)}\Big[\mathbb{R}\times\cdots\times\mathbb{R}\times\{x+k_{\ell_1}\}\times\mathbb{R}\times\cdots\times\mathbb{R}\times\{x+k_{\ell_n}\}\times\mathbb{R}\times\cdots\times\mathbb{R}\Big],
\end{aligned}
$$

so that the finite-dimensional marginals are consistent. Therefore, the Kolmogorov consistency theorem ensures that there exists a probability measure P on $(S^{[0,\infty)},\mathcal{S}^{[0,\infty)})$ which is consistent with all the finite-dimensional marginals $\nu^{(I)}$. It follows that $(Y_t)_{t\ge0}$ has independent increments, with $P[Y_0=x]\ =1$ and $Y_t-Y_s\sim\mathrm{Poi}(\lambda(t-s))$, since these are statements about the finite dimensional marginals that follow immediately by construction.

(b) By construction and countability, we see that the process $(Y_q)_{q\in\mathbb{Q}_+}$ is (outside of a P-nullset) an increasing piecewise constant process with $Y_0=x$ and all jumps $\in\mathbb{N}$.

Next, we show that

$$
P\left[Y_t=\lim_{\mathbb{Q}\ni q\nearrow t}Y_q=\lim_{\mathbb{Q}\ni q\searrow t}Y_q\right]=1
$$

for each $t\in\mathbb{R}_+$. Indeed, $P[Y_t\ge Y_q]=1$ for all $q<t$ by construction, so $Y_t\ge\sup_{\mathbb{Q}\ni q<t}Y_q$ a.s.; on the other hand $P[Y_t>Y_q]=1-\exp(-\lambda(t-q))\searrow0$ as $q\nearrow t$, so

$$
P[Y_t\ge 1+\sup_{\mathbb{Q}\ni q<t}Y_q]\le\lim_{\mathbb{Q}\ni q\nearrow t}P[Y_t\ge Y_q+1]=0.
$$

The other side is analogous.

We can also show that $(Y_q)_{q\in\mathbb{Q}_+}$ only has jumps of size $+1$. Indeed, for all $m\in\mathbb{N}$,

$$
\begin{aligned}
P\big[(Y_q)_{q\in\mathbb{Q}\cap[0,m]}\text{ has a jump of size }>1\big] &\le \inf_{n\in\mathbb{N}}P\left[\bigcup_{j=1}^{m2^n}\{Y_{2^{-n}j}-Y_{2^{-n}(j-1)}\ge2\}\right]\\
&\le \inf_{n\in\mathbb{N}}\sum_{j=1}^{m2^n}P\left[Y_{2^{-n}j}-Y_{2^{-n}(j-1)}\ge2\right]\\
&= \inf_{n\in\mathbb{N}}m2^n\sum_{k=2}^{\infty}e^{-\lambda2^{-n}}\frac{(\lambda2^{-n})^k}{k!}=0.
\end{aligned}
$$

Taking a union over $m\in\mathbb{N}$ we extend the result to all of $\mathbb{Q}_+$.

Now, construct the process $(N_t)_{t\ge0}$ by $N_t=\lim_{\mathbb{Q}\ni q\searrow t}Y_q$. Since $(Y_q)_{q\in\mathbb{Q}}$ is a.s. an increasing piecewise constant process with all jumps of size $+1$, the same must hold for $(N_t)_{t\ge0}$, and moreover N is RCLL. We have that N is a version of Y since $P[Y_t = \lim_{q \searrow t} Y_q = N_t] = 1$ for all $t \geq 0$, and therefore $N_0 = x$ and the increments $N_t - N_s \sim \mathrm{Poi}(\lambda(t-s))$ are independent, since the same holds for Y.

(c) For $t \geq 0$, we have that

$$P[\tau_n > t] = P[N_t \leq n-1] = \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

Differentiating in t,

$$\begin{aligned} f_{\tau_n}(t) &= -\sum_{k=0}^{n-1} \lambda e^{-\lambda t} \left(-\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} + \sum_{k=1}^{n-1} \frac{k(\lambda t)^{k-1}}{k!} \right) \\ &= -\lambda e^{-\lambda t} \left(-\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} + \sum_{k=0}^{n-2} \frac{(\lambda t)^k}{k!} \right) \\ &= \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \end{aligned}$$

therefore $\tau_n \sim \mathrm{Gamma}(n, \lambda)$.

(d) This follows from stationary independent increments property, since for $s \leq t$ and measurable bounded $f : \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} E[f(N_t) \mid \sigma(N_r : r \leq s)] &= E\big[f\big(N_s + (N_t - N_s)\big) \bigm| \sigma(N_r : r \leq s)\big] \\ &= \sum_{k=0}^{\infty} f(N_s + k) e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!} \end{aligned}$$

which implies that N is Markov with kernel

$$K_h(x, A) = \sum_{k=0}^{\infty} \delta_{x+k}(A) e^{-\lambda h} \frac{(\lambda h)^k}{k!}.$$

With the same argument, Y is Markov as well with the same kernel.