Brownian Motion and Stochastic Calculus

Exercise sheet 6

Exercise 6.1 Let $(S,\mathcal{S}) = (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and for each $x \in \mathbb{R}^2$, let $\mathbb{P}_x$ denote the unique probability measure on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ under which the coordinate process Y is a 2-dimensional Brownian motion starting at x. Show that for any $x \in \mathbb{R}^2$,

$$\mathbb{P}_x\left[\sup\{t \geq 0 : Y_t \in \mathcal{O}\} = \infty \text{ for every non-empty open set } \mathcal{O} \subseteq \mathbb{R}^2\right] = 1.$$

Hint: For any $x \in \mathbb{R}^2$ and $r \geq 0$, we define $\overline{B}(x,r) := \{y \in \mathbb{R}^2 : |x-y| \leq r\}$ and the stopping time $T_{\overline{B}(x,r)} := \inf\{t \geq 0 : Y_t \in \overline{B}(x,r)\}$. Use the fact that for any $x \in \mathbb{R}^2$ and $r > 0$, we have $T_{\overline{B}(0,r)} < \infty$ $\mathbb{P}_x$-a.s., and apply the strong Markov property of Brownian motion.
Remark: This exercise shows the *recurrence of Brownian motion in* $\mathbb{R}^2$.

Solution 6.1 From the given hint, we know that

$$\text{for any } x \in \mathbb{R}^2 \text{ and } r > 0, \text{ we have } T_{\overline{B}(0,r)} < \infty \ \ \mathbb{P}_x\text{-a.s.} \tag{1}$$

Moreover, $T_{\overline{B}(0,r)}$ is a $\mathbb{Y}$-stopping time. Let $r > 0$. We define the sequence of $\mathbb{Y}$-stopping times (T_i) by

$$\begin{aligned} T_1 &= T_{\overline{B}(0,r)}, \\ T_{i+1} &= T_1 \circ \vartheta_{T_i+1} + T_i + 1 \\ &= \inf\{t \geq T_i + 1 : Y_t \in \overline{B}(0,r)\} \qquad \text{for } i \geq 1. \end{aligned}$$

Thus, (T_i) converges strictly monotonically to infinity. Using the strong Markov property in the fourth equality, (**??**) in the sixth and then induction, we see that for any $y \in \mathbb{R}^2$, for $i \geq 1$,

$$\begin{aligned} \mathbb{P}_y\big[T_{i+1} < \infty\big] &= \mathbb{E}_y\Big[\mathbf{1}_{\{T_1 \circ \vartheta_{T_i+1} + T_i + 1 < \infty\}}\Big] = \mathbb{E}_y\Big[\big(\mathbf{1}_{\{T_1<\infty\}} \circ \vartheta_{T_i+1}\big)\, \mathbf{1}_{\{T_i<\infty\}}\Big] \\ &= \mathbb{E}_y\Big[\mathbf{1}_{\{T_i<\infty\}}\, \mathbb{E}_y\Big[\mathbf{1}_{\{T_1<\infty\}} \circ \vartheta_{T_i+1} \,\Big|\, \mathcal{Y}_{T_i+1}\Big]\Big] = \mathbb{E}_y\Big[\mathbf{1}_{\{T_i<\infty\}}\, \mathbb{E}_{Y_{T_i+1}}\Big[\mathbf{1}_{\{T_1<\infty\}}\Big]\Big] \\ &= \mathbb{E}_y\Big[\mathbf{1}_{\{T_i<\infty\}}\, \mathbb{P}_{Y_{T_i+1}}\Big[T_1 < \infty\Big]\Big] = \mathbb{P}_y\big[T_i < \infty\big] = \mathbb{P}_y\big[T_1 < \infty\big] \\ &= 1. \end{aligned}$$

Note also that by construction, for $i \geq 1$, we have $Y_{T_i} \in \overline{B}(0,r)$ $\mathbb{P}_y$-a.s. on $\{T_i < \infty\}$. It thus follows that for any $y \in \mathbb{R}^2$, the set $\{t \geq 0 : Y_t \in \overline{B}(0,\frac{1}{n})\}$ is unbounded $\mathbb{P}_y$-a.s. for any $r = \frac{1}{n}$, $n \in \mathbb{N}$. Now, since $\mathbb{P}_y$ is the law of $(y+Y_t)_{t\geq 0}$ under $\mathbb{P}_0$, the above property implies that

$$\mathbb{P}_0\text{-a.s., the set } \{t \geq 0 : Y_t \in \overline{B}(z,\tfrac{1}{n})\} \text{ is unbounded for all } z \in \mathbb{Q}^2, n \geq 1.$$

This proves the claim for $x = 0$ as for every open set $\mathcal{O} \subseteq \mathbb{R}^2$, we can find $z \in \mathbb{Q}^2$ and $n \geq 1$ such that $\overline{B}(z,\frac{1}{n}) \subseteq \mathcal{O}$. The case of general $x \in \mathbb{R}^2$ follows since $(Y_t)_{t\geq 0}$ under $\mathbb{P}_x$ has the same law as $(x+Y_t)_{t\geq 0}$ under $\mathbb{P}_0$ and $\mathcal{O} - x \subseteq \mathbb{R}^2$ is open whenever $\mathcal{O}$ is open.

Exercise 6.2

(a) Let $(Z_t)_{t\geq 0}$ be an adapted process with respect to a given filtration $(\mathcal{F}_t)_{t\geq 0}$ such that for every bounded continuous function f, we have

$$E\big[f(Z_t - Z_s)\,\big|\,\mathcal{F}_s\big] = E\big[f(Z_{t-s})\big].$$

Show that Z has stationary independent increments.

(b) Let W be a Brownian motion on $(\Omega, \mathcal{F}, P)$. For every $a \geq 0$, consider the entrance time

$$T_a := \inf\{s \geq 0 : W_s \geq a\}.$$

Show that the process $(T_a)_{a\geq 0}$ has stationary independent increments.

(c) Let $(Z_t)_{t\geq 0}$ have stationary independent increments and start at 0, (W_t) be a Brownian motion independent of (Z_t), and $(T_a)_{a\geq 0}$ as in (b). Show that $(\hat{Z}_t)_{t\geq 0} = (Z_{T_t})_{t\geq 0}$ has stationary independent increments.

Remark: The process $(T_t)_{t\geq 0}$ is called a subordinator.

Solution 6.2

(a) We show that for times $0 \leq t_0 < t_1 < \cdots < t_n$ and measurable bounded functions $f^i : \mathbb{R} \to \mathbb{R}\,(i = 1, \ldots, n)$, we have that

$$E\left[\prod_{i=1}^n f^i(Z_{t_i} - Z_{t_{i-1}})\right] = \prod_{i=1}^n E[f^i(Z_{t_i - t_{i-1}})].$$

If the f^i are bounded and continuous, we have by assumption that

$$\begin{aligned}
E\left[\prod_{i=1}^n f^i(Z_{t_i} - Z_{t_{i-1}})\right] &= E\left[E\left[\prod_{i=1}^n f^i(Z_{t_i} - Z_{t_{i-1}}) \,\middle|\, \mathcal{F}_{t_{n-1}}\right]\right] \\
&= E\left[\prod_{i=1}^{n-1} f^i(Z_{t_i} - Z_{t_{i-1}})\right] E[f^n(Z_{t_n - t_{n-1}})] \\
&= \prod_{i=1}^n E[f^i(Z_{t_i - t_{i-1}})],
\end{aligned}$$

proceeding by induction.

If $f^i = \mathbb{1}_{A_i}$ for $i = 1, \ldots, n$ and some $A_i \in \mathcal{B}(\mathbb{R})$, we can find continuous $f^{i,m}$ with a uniform bound $\sup_{m,i}[\operatorname{ess\,sup} f^{i,m}] \leq 1$ such that $f^{i,m} \to f^i$ pointwise on $\mathbb{R}$. Therefore, we find by two applications of the dominated convergence theorem that

$$\begin{aligned}
E\left[\prod_{i=1}^n f^i(Z_{t_i} - Z_{t_{i-1}})\right] &= \lim_{m\to\infty} E\left[\prod_{i=1}^n f^{i,m}(Z_{t_i} - Z_{t_{i-1}})\right] \\
&= \lim_{m\to\infty} \prod_{i=1}^n E[f^{i,m}(Z_{t_i - t_{i-1}})] = \prod_{i=1}^n E[f^i(Z_{t_i - t_{i-1}})].
\end{aligned}$$

This extends to simple functions by linearity, and then to measurable bounded functions by approximation with simple functions. This proves the claim.

This already establishes independence, and we now show that

$$E\left[f(Z_{t_1} - Z_{t_0}, \ldots, Z_{t_n} - Z_{t_{n-1}})\right] = E\left[f(Z_{t_1+h} - Z_{t_0+h}, \ldots, Z_{t_n+h} - Z_{t_{n-1}+h})\right]$$

for any $h \geq 0$ and bounded measurable $f : \mathbb{R}^k \to \mathbb{R}$, which implies stationarity of the increments. Letting $\mathcal{H}$ be the set of bounded measurable functions such that this is satisfied for all $h \geq 0$, we have that $\mathcal{H}$ is a vector space, contains 1 and is closed under bounded monotone convergence by the dominated convergence theorem. Moreover, by what we showed above, $\mathcal{H} \supseteq \mathcal{M}$, where $\mathcal{M}$ is the set of functions of the form $f(z_1, \ldots, z_n) = \prod_{i=1}^n f^i(z_i)$ for bounded measurable f^i, since

$$E\left[\prod_{i=1}^n f^i(Z_{t_i} - Z_{t_{i-1}})\right] = \prod_{i=1}^n E\left[f^i\left(Z_{t_i - t_{i-1}}\right)\right] = E\left[\prod_{i=1}^n f^i(Z_{t_i+h} - Z_{t_{i-1}+h})\right]$$

for all $h \geq 0$. Using the monotone class theorem, we conclude that $\mathcal{H}$ contains all bounded measurable functions and therefore Z has stationary increments.

(b) Since the claim refers only to distributional properties of $(W_t)_{t\geq 0}$, we may assume w.l.o.g. that $(W_t) = (Y_t)$ is given as a coordinate process on $S^{[0,\infty)}$ with measure $\mathbb{P}_0$. We first notice that if $0 < a < b$, then

$$T_b = \inf\{s \geq 0 : Y_s \geq b\} = T_a + \inf\{s \geq 0 : Y_{T_a+s} \geq b\} = T_a + T_b \circ \vartheta_{T_a}.$$

So for every bounded measurable function f, by the strong Markov property of Brownian motion and as $(s + Y_u)_{u\geq 0}$ has the same law under $\mathbb{P}_0$ as $(Y_u)_{u\geq 0}$ under $\mathbb{P}_s$, we obtain that

$$\mathbb{E}_0\big[f(T_b - T_a) \,\big|\, \mathcal{Y}_{T_a}\big] = \mathbb{E}_0\big[f(T_b) \circ \vartheta_{T_a} \,\big|\, \mathcal{Y}_{T_a}\big] = \mathbb{E}_{Y_{T_a}}\big[f(T_b)\big] = \mathbb{E}_a\big[f(T_b)\big] = \mathbb{E}_0\big[f(T_{b-a})\big],$$

where $\mathbb{P}_a$ denotes the law of $(a+Y_t)_{t\geq 0}$ under $\mathbb{P}_0$. Since the process $(T_a)_{a\geq 0}$ is clearly adapted to the filtration $(\mathcal{Y}_{T_a})_{a\geq 0}$, we obtain the desired result directly from (a).

(c) Let $0 = t_0 \leq t_1 \leq \cdots \leq t_n$ and $f^i : \mathbb{R} \to \mathbb{R}$ be bounded and measurable. Using the independence properties as well as (a) and (b), we obtain

$$\begin{aligned}
E\left[\prod_{i=1}^n f^i(\hat{Z}_{t_i} - \hat{Z}_{t_{i-1}})\right] &= E\left[\prod_{i=1}^n f^i(Z_{\tau_{t_i}} - Z_{\tau_{t_{i-1}}})\right] \\
&= E\left[E\left[\prod_{i=1}^n f^i(Z_{\tau_{t_i}} - Z_{\tau_{t_{i-1}}}) \,\middle|\, \mathcal{F}^W_\infty\right]\right] \\
&= E\left[E\left[\prod_{i=1}^n f^i(Z_{s_i} - Z_{s_{i-1}})\right]\Big|_{s_i=\tau_{t_i}}\right] \\
&= E\left[\prod_{i=1}^n E\left[f^i(Z_{s_i} - Z_{s_{i-1}})\right]\big|_{s_i=\tau_{t_i}}\right] \\
&= \prod_{i=1}^n E\big[f^i(Z_{(\tau_{t_i} - \tau_{t_{i-1}})})\big] \\
&= \prod_{i=1}^n E\big[E[f^i(Z_{(\tau_{t_i} - \tau_{t_{i-1}})}) \mid \mathcal{F}^Z_\infty]\big] \\
&= \prod_{i=1}^n E\big[f^i(Z_{\tau_{t_i - t_{i-1}}})\big] \\
&= \prod_{i=1}^n E\big[f^i(\hat{Z}_{t_i - t_{i-1}})\big].
\end{aligned}$$

By the same argument as in (a), this implies that $\hat{Z}$ has independent and stationary increments.

Exercise 6.3 Let $\phi : \mathbb{R}^k \to \mathbb{R}^k$ be a locally Lipschitz function with linear growth, meaning that

$$\begin{aligned} |\phi(x) - \phi(y)| &\le C_n|x-y| \quad \text{for some } C_n \ge 0 \text{ and all } x, y : |x|, |y| \le n, \text{ and} \\ |\phi(x)| &\le C(1+|x|) \quad \text{for some } C \ge 0. \end{aligned}$$

For each $x \in \mathbb{R}^k$, define $X^x : [0,\infty) \to \mathbb{R}^k$ as the solution to the ODE

$$\begin{cases} \dfrac{dX_t^x}{dt} = \phi(X_t^x), \quad t \ge 0, \\ X_0^x = x. \end{cases} \tag{2}$$

Due the assumptions on ϕ, each ODE has a unique solution X^x (you do not need to prove this).

(a) Define the unique probability measures $\mathbb{P}_x$ on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$ such that

$$\mathbb{P}_x[Y_t = X_t^x] = 1$$

for all $x \in \mathbb{R}^k$ and $t \ge 0$. Show that Y is a strong Markov process.

(b) Construct an example where $\phi : \mathbb{R}^k \to \mathbb{R}^k$ is a continuous function and (X^x) is a solution to (1) for each x, but the measures

$$\mathbb{P}_x[Y_t = X_t^x] = 1$$

do not define a Markov process.

Solution 6.3

(a) Since ϕ is locally Lipschitz with linear growth, the solutions X^x are unique and continuously differentiable on $[0,\infty)$. We first note that for $s, t \ge 0$,

$$X_{s+t}^x = X_t^{X_s^x}.$$

This follows from the fact that $(X_{s+t}^x)_{t\ge 0}$ is the unique solution to (1) with $X_{s+0}^x = X_s^x$.

Now, we define

$$\mathbb{P}_x\big[(Y_{t_1}, \dots, Y_{t_n}) \in A\big] = \delta_{(X_{t_1}^x, \dots, X_{t_n}^x)}(A)$$

for $x \in \mathbb{R}^k$, $A \subseteq \mathbb{R}^{k\times n}$ and $0 \le t_1 \le \dots \le t_n$, which defines via Kolmogorov's consistency theorem a unique probability measure on $(S^{[0,\infty)}, \mathcal{S}^{[0,\infty)})$.

Since $\mathbb{P}_x$ is a point mass, any random variable is constant up to a nullset, and $\mathcal{S}^{[0,\infty)}$ is $\mathbb{P}_x$-trivial for all $x \in \mathbb{R}$. Concretely, for any random variable $Z : S^{[0,\infty)} \to \mathbb{R}$, we have that $Z((Y_t)_{t\ge 0}) = Z((X_t^x)_{t\ge 0})$ $\mathbb{P}_x$-a.s. for each $x \in \mathbb{R}^k$. Therefore, both the Markov and strong Markov property are in this case equivalent to the deterministic identity

$$\vartheta_s\big((X_t^x)_{t\ge 0}\big) = (X_t^{X_s^x})_{t\ge 0},$$

which we showed above.

(b) Consider $k = 1$ and let $\phi(y) = 2\sqrt{|y|}$. Note that ϕ is not locally Lipschitz; in particular, (1) does not have a unique solution. In fact, we can find solutions

$$X_t^x = \begin{cases} \left(\sqrt{x} + t\right)^2, & x > 0, \\ 0, & x = 0, \\ \left|t - \sqrt{|x|}\right| \left(t - \sqrt{|x|}\right), & x < 0 \end{cases}$$

to (1). However, we note that for $x < 0$,

$$\vartheta_{\sqrt{|x|}}\big((X_t^x)_{t\ge 0}\big) = (t^2)_{t\ge 0},$$

whereas

$$X^{X_{\sqrt{|x|}}^x} \equiv X^0 \equiv 0,$$

and therefore the associated process Y is not Markov.