Brownian Motion and Stochastic Calculus

Exercise sheet 7

Exercise 7.1 Let $(W_t)_{t\geq 0}$ be a 2-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ started at 0 and $C \neq \emptyset$ an open cone in $\mathbb{R}^2$, i.e. C is an open set and for every $x \in C$, we have $\lambda x \in C$ for all $\lambda > 0$. Note that 0 need not belong to C. Consider the hitting time T_C^W of C, i.e.

$$T_C^W := \inf \big\{t > 0 : W_t \in C\big\}.$$

Show that $T_C^W = 0$ P-a.s.

Solution 7.1 Let $\mathbb{P}_0 = P \circ W^{-1}$ be Wiener measure on $C([0,\infty);\mathbb{R})$ and $(Y_t)_{t\geq 0}$ the coordinate process. Define

$$T_C^Y := \inf\{t > 0 : Y_t \in C\}.$$

We first show that $\{T_C^Y = 0\} \in \mathcal{Y}_{0+}^0 = \mathcal{Y}_0$. Indeed, using that C is open, this follows directly from the identity

$$\big\{T_C^Y = 0\big\} = \bigcap_{n=1}^{\infty} \Big(\bigcup_{r\in(0,\frac{1}{n}]\cap\mathbb{Q}} \{Y_r \in C\}\Big) \in \mathcal{Y}_0,$$

since $B_n = \bigcup_{r\in(0,\frac{1}{n}]\cap\mathbb{Q}} \{Y_r \in C\} \in \mathcal{Y}_{1/n}^0$ and $B_1 \supseteq B_2 \supseteq \cdots$. Recall the scaling property of Brownian motion, i.e. that for any $t \geq 0$,

$$Y_t \overset{(d)}{=} \sqrt{t}Y_1$$

under $\mathbb{P}_0$. Using this property and that C is a cone, we obtain for every $t > 0$ that

$$\mathbb{P}_0[T_C^Y \leq t] \geq \mathbb{P}_0[Y_t \in C] = \mathbb{P}_0[\sqrt{t}Y_1 \in C] = \mathbb{P}_0[Y_1 \in C] > 0,$$

where the last inequality holds true since C has strictly positive Lebesgue measure and Y_1 is bivariate normally distributed. Thus we obtain that

$$\mathbb{P}_0[T_C^Y = 0] = \lim_{t\to 0} \mathbb{P}_0[T_C^Y \leq t] \geq \mathbb{P}_0[Y_1 \in C] > 0$$

and therefore, by the Blumenthal 0-1 law, $\mathbb{P}_0[T_C^Y = 0] = 1$. Since T_C^W has the same law under P as T_C^Y under $\mathbb{P}_0$, we conclude that

$$P[T_C^W = 0] = \mathbb{P}_0[T_C^Y = 0] = 1.$$

Exercise 7.2 Let $\Omega = C([0,\infty);\mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ denote the coordinate process. For each $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the unique probability measure on $(\Omega, \mathcal{Y}^0_\infty)$ under which Y is a (d-dimensional) Brownian motion started at x. Moreover, for any open set $A \subseteq \mathbb{R}^d$, we denote by

$$\tau_A := \inf\{t \geq 0 : Y_t \notin A\}$$

the first exit time of the Brownian motion Y from the set A.

Fix an open set $G \subseteq \mathbb{R}^d$ such that $\mathbb{E}_x[\tau_G] < \infty$ for all $x \in G$, a bounded Borel function $g : \mathbb{R}^d \to \mathbb{R}$, and define the function $u : G \to \mathbb{R}$ by

$$u(y) := \mathbb{E}_y\left[\int_0^{\tau_G} g(Y_s)\,ds\right].$$

Moreover, for any $\varepsilon > 0$ and $x \in \mathbb{R}^d$, we let $U_\varepsilon(x) := \{y : |y - x| < \varepsilon\}$ denote the open ε-ball around x and set $\sigma_\varepsilon(x) := \tau_{U_\varepsilon(x)}$.

Fix $\varepsilon > 0$ and $x \in G$ such that $U_\varepsilon(x) \subseteq G$. Show that

$$u(x) = \mathbb{E}_x\left[u\left(Y_{\sigma_\varepsilon(x)}\right) + \int_0^{\sigma_\varepsilon(x)} g(Y_s)\,ds\right].$$

Hint: First show that $\tau_G = \tau_G \circ \vartheta_{\sigma_\varepsilon(x)} + \sigma_\varepsilon(x)$. Then compute $u(x)$ by conditioning on $\mathcal{F}_{\sigma_\varepsilon(x)}$ and using the strong Markov property.

Solution 7.2 We first observe that

$$\tau_G = \inf\big\{t \geq 0 \,:\, Y_t \in G^c\big\}.$$

So if G is open, G^c is closed and thus one can show that τ_G is a $\mathbb{Y}$-stopping time. Fix $\varepsilon > 0$ and $x \in G$ such that $U_\varepsilon(x) \subseteq G$. Then $\sigma_\varepsilon(x) \leq \tau_G$. Hence,

$$\begin{aligned}\tau_G \circ \vartheta_{\sigma_\varepsilon(x)} + \sigma_\varepsilon(x) &= \inf\{t \geq 0 : Y_t \circ \vartheta_{\sigma_\varepsilon(x)} \notin G\} + \sigma_\varepsilon(x) \\ &= \inf\{t \geq 0 : Y_{t+\sigma_\varepsilon(x)} \notin G\} + \sigma_\varepsilon(x) = \tau_G. \end{aligned} \tag{1}$$

Note that the process $\int_0^\cdot g(Y_s)\,ds$ is continuous and adapted. Thus, $\int_0^{\sigma_\varepsilon(x)} g(Y_s)\,ds$ is $\mathcal{Y}_{\sigma_\varepsilon(x)}$-measurable. Conditioning on $\mathcal{Y}_{\sigma_\varepsilon(x)}$ yields

$$u(x) = \mathbb{E}_x\left[\int_0^{\tau_G} g(Y_s)\,ds\right] = \mathbb{E}_x\left[\int_0^{\sigma_\varepsilon(x)} g(Y_s)\,ds + \mathbb{E}_x\Big[\int_{\sigma_\varepsilon(x)}^{\tau_G} g(Y_s)\,ds\,\Big|\,\mathcal{Y}_{\sigma_\varepsilon(x)}\Big]\right]. \tag{2}$$

To compute the conditional expectation on the right-hand side in (2), using (1) we note that

$$\int_{\sigma_\varepsilon(x)}^{\tau_G} g(Y_s)\,ds = \int_0^{\tau_G - \sigma_\varepsilon(x)} g(Y_{s+\sigma_\varepsilon(x)})\,ds = \left(\int_0^{\tau_G} g(Y_s)\,ds\right) \circ \vartheta_{\sigma_\varepsilon(x)}. \tag{3}$$

The strong Markov property and (3) then give that

$$\begin{aligned}\mathbb{E}_x\left[\int_{\sigma_\varepsilon(x)}^{\tau_G} g(Y_s)\,ds\,\middle|\,\mathcal{Y}_{\sigma_\varepsilon(x)}\right] &= \mathbb{E}_x\left[\left(\int_0^{\tau_G} g(Y_s)\,ds\right) \circ \vartheta_{\sigma_\varepsilon(x)}\,\middle|\,\mathcal{Y}_{\sigma_\varepsilon(x)}\right] \\ &= \mathbb{E}_{Y_{\sigma_\varepsilon(x)}}\left[\int_0^{\tau_G} g(Y_s)\,ds\right] = u(Y_{\sigma_\varepsilon(x)}), \quad \mathbb{P}_x\text{-a.s.}\end{aligned} \tag{4}$$

Finally, inserting (4) into (2) yields the desired result.

Exercise 7.3 Assume we have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions.

(a) Let $M \in \mathcal{M}^c_{0,\text{loc}}$. Prove that $M \in \mathcal{H}^{2,c}_0$ if and only if $E[\langle M \rangle_\infty] < \infty$, and that in this case $\|M\|^2_{\mathcal{H}^2} = E[\langle M \rangle_\infty]$.

(b) An optional process X is said to be of class (DL) if for all $a > 0$, the family

$$\mathfrak{X}_a := \big\{ X_\tau : \tau \text{ stopping time, } \tau \le a \ P\text{-a.s.} \big\}$$

is uniformly integrable. Show that a local martingale null at 0 is a (true) martingale null at 0 if and only if it is of class (DL).

Remarks:

- As a consequence, we obtain that a local martingale M null at 0 and with integrable supremum, i.e. $M^*_t := \sup_{0 \le s \le t} |M_s| \in L^1(P)$ for all $t \ge 0$, is a true martingale.
- There exist local martingales null at 0 which are uniformly integrable (i.e. the family $\{M_t : t \ge 0\}$ is uniformly integrable), but are not true martingales.

Solution 7.3

(a) For $M \in \mathcal{M}^c_{0,\text{loc}}$, let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence of stopping times for M such that $(M^2 - \langle M \rangle)^{\tau_n} \in \mathcal{M}^c_0$. We then have that

$$E[M^2_{\tau_n \wedge t}] = E[\langle M \rangle_{\tau_n \wedge t}], \qquad \text{for each } n \in \mathbb{N}, t \ge 0. \tag{5}$$

We show the first implication. Let $M \in \mathcal{H}^{2,c}_0$; then by the martingale convergence theorem, $M_t \to M_\infty \in L^2$ a.s. and in L^2. Moreover, Doob's inequality gives $M^*_\infty = \sup_{t \ge 0} |M_t| \in L^2$. We can therefore apply the dominated convergence theorem on the left-hand side of (5) for $n \to \infty$. On the other hand, $\langle M \rangle$ is increasing, so that the monotone convergence theorem applied on the right-hand side of (5) for $n \to \infty$ gives

$$E[M^2_t] = E[\langle M \rangle_t] \quad \forall\, t \ge 0.$$

By applying the dominated and the monotone convergence theorem again for $t \to \infty$, we conclude that

$$E[M^2_\infty] = E[\langle M \rangle_\infty] < \infty.$$

Conversely, assume that $E[\langle M \rangle_\infty] < \infty$. By using (5) and the fact that $\langle M \rangle$ is increasing, we obtain that

$$E[M^2_{\tau_n \wedge t}] = E[\langle M \rangle_{\tau_n \wedge t}] \le E[\langle M \rangle_\infty] =: K < \infty. \tag{6}$$

By applying Fatou's lemma to (6), we obtain that

$$E[M^2_t] \le \liminf_{n \to \infty} E[M^2_{\tau_n \wedge t}] \le K < \infty,$$

so that M is bounded in L^2. Moreover, by (6), $(M_{\tau_n \wedge t})_{n \in \mathbb{N}}$ is bounded in L^2 and hence uniformly integrable. Next, we want to pass to the limit in the equality

$$E[M_{\tau_n \wedge t} | \mathcal{F}_s] = M_{\tau_n \wedge s} \quad \forall\, n \in \mathbb{N}.$$

To do that, let $A \in \mathcal{F}_s$ and note that $E[M_{\tau_n \wedge s} \mathbb{1}_A] \to E[M_s \mathbb{1}_A]$ as $n \to \infty$ by the dominated convergence theorem, and likewise $E[M_{\tau_n \wedge t} \mathbb{1}_A] \to E[M_t \mathbb{1}_A]$ as $n \to \infty$. Therefore, we obtain that $E[M_t | \mathcal{F}_s] = M_s$, so that M is a martingale, and hence in $\mathcal{H}^{2,c}_0$.

(b) Let M be a true martingale and fix $a > 0$. Then the stopping theorem implies

$$M_\tau = E[M_a|\mathcal{F}_\tau]$$

for all stopping times τ with $\tau \leq a$ P-a.s. Since $M_a \in L^1$, we then obtain that $\mathfrak{X}_a$ is uniformly integrable. More specifically, for any $X \in L^1$, if holds that $\{E[X \mid \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F} \text{ a } \sigma\text{-algebra}\}$ is uniformly integrable. One proof is as follows: by the de la Vallée–Poussin theorem, since $X \in L^1$ and so the family $\{X\}$ is uniformly integrable, there exists a non-negative increasing convex function φ with $\lim_{x\to\infty} \varphi(x)/x = \infty$ such that $E[\varphi(|X|)] = C < \infty$. By Jensen's inequality, for any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$,

$$E\Big[\varphi\big(\big|E[X \mid \mathcal{G}]\big|\big)\Big] \leq E\big[E[\varphi(|X|) \mid \mathcal{G}]\big] = E\big[\varphi(|X|)\big] \leq C.$$

Therefore by de la Vallée–Poussin theorem again, we have that $\{E[X \mid \mathcal{G}] : \mathcal{G} \subseteq F \text{ a } \sigma\text{-algebra}\}$ is uniformly integrable. In particular, the subfamily $\{M_\tau = E[M_a|\mathcal{F}_\tau] : \tau \leq a \text{ a stopping time}\}$ is uniformly integrable.

Conversely, assume we have a local martingale M of class (DL), and let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence. Since $\mathfrak{X}_t$ is uniformly integrable for all $t \geq 0$, we obtain that $\{M_{\tau_n \wedge t} : n \in \mathbb{N}\}$ is also uniformly integrable. Therefore, the fact that $M_{\tau_n\wedge t} \overset{n\to\infty}{\longrightarrow} M_t$ P-a.s. implies that

$$M_{\tau_n \wedge t} \overset{n\to\infty}{\longrightarrow} M_t \text{ in } L^1$$

and also $M_t \in L^1$. As in (a), we can then pass to the limit in the equality $E[M_{\tau_n\wedge t}|\mathcal{F}_s] = M_{\tau_n\wedge s}$, obtaining that $E[M_t|\mathcal{F}_s] = M_s$, so that M is a martingale.

Exercise 7.4 For any function $f : [0,\infty) \to \mathbb{R}$ with $f(0) = 0$, we define its total variation (or 1-variation) $|f| : [0,\infty) \to [0,\infty]$ by

$$\begin{aligned}|f|(t) &:= \sup\left\{ \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| : \Pi \text{ is a partition of } [0,t]\right\}\\ &= \sup\left\{ \sum_{t_i \in \Pi} \big|f(t_{i+1} \wedge t) - f(t_i \wedge t)\big| : \Pi \text{ is a partition of } [0,\infty)\right\}.\end{aligned}$$

We say that f has finite variation (FV) if $|f|(t) < \infty$ for all $t \geq 0$.

(a) Show that f has finite variation if and only if there exist two non-decreasing functions $f_1, f_2 : [0,\infty) \to \mathbb{R}$ with $f_1(0) = f_2(0) = 0$ such that $f = f_1 - f_2$.

If so, find the minimal such functions f_1 and f_2, in the sense that $\tilde{f}_1 \geq f_1$ and $\tilde{f}_2 \geq f_2$ for any other non-decreasing functions $\tilde{f}_1, \tilde{f}_2$ with $\tilde{f}_1(0) = \tilde{f}_2(0) = 0$ such that $f = \tilde{f}_1 - \tilde{f}_2$.

Hint: Start by showing that $|f|(t) - |f|(s) \geq |f(t) - f(s)|$ for $0 \leq s \leq t$.

(b) Show that if f is right-continuous and has finite variation, then $|f|$ is right-continuous.

Using the Carathéodory extension theorem, one can show that for any non-decreasing right-continuous function $\tilde{f}$, there exists a unique positive measure $\mu_{\tilde{f}}$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\mu_{\tilde{f}}((0,t]) = \tilde{f}(t) - \tilde{f}(0)$ for all $t \geq 0$.

(c) Let f be right-continuous with finite variation with $f(0) = 0$ and $g : [0,\infty) \to \mathbb{R}$ such that

$$\int_0^\infty |g(s)|\, \mu_{|f|}(ds) < \infty.$$

Let f_1, f_2 be the minimal functions defined in (a). Show that

$$\int_0^\infty |g(s)|\, d\mu_{f_1}(s) < \infty, \quad \int_0^\infty |g(s)|\, d\mu_{f_2}(s) < \infty,$$

so that

$$\int g(s)\, df(s) := \int g(s)\, d\mu_{f_1}(s) - \int g(s)\, d\mu_{f_2}(s)$$

is well defined.

Remark: If f is of finite variation and right-continuous, a function g is *f-integrable in the Lebesgue–Stieltjes sense* if g satisfies $\int_0^\infty |g(s)|\, \mu_{|f|}(ds) < \infty$. In that case, we define the *Lebesgue–Stieltjes integral* to be $\int g(s)\, df(s)$.

Solution 7.4

(a) For the "if" direction, let $f = f_1 - f_2$, where f_1 and f_2 are non-decreasing. Then we have for any $t \geq 0$ and any partition Π of $[0,t]$ that

$$\begin{aligned}\sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| &= \sum_{t_i \in \Pi} \Big|\big(f_1(t_{i+1}) - f_2(t_{i+1})\big) - \big(f_1(t_i) - f_2(t_i)\big)\Big|\\ &\leq \sum_{t_i \in \Pi} \big|f_1(t_{i+1}) - f_1(t_i)\big| + \sum_{t_i \in \Pi} \big|f_2(t_{i+1}) - f_2(t_i)\big|\\ &\leq f_1(t) + f_2(t).\end{aligned}$$

Thus,

$$|f|(t) = \sup_{\Pi_{[0,t]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \leq f_1(t) + f_2(t) < \infty \quad \text{for every } t \geq 0.$$

For the "only if" direction, let $f : [0,\infty) \to \mathbb{R}$ have finite variation. Define

$$f_1(t) := \frac{|f|(t) + f(t)}{2} \quad \text{and} \quad f_2(t) := \frac{|f|(t) - f(t)}{2}.$$

We claim that both f_1, f_2 are non-decreasing. To show that, fix $0 \leq s < t$ and denote by Π_I the set of partitions of an interval I, where I can be any of $[0,s], [0,t]$ or $[s,t]$. Then,

$$\begin{aligned}
|f|(s) &= \sup_{\Pi_{[0,s]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| \\
&= \sup_{\Pi_{[0,s]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| + |f(t) - f(s)| - |f(t) - f(s)| \\
&\leq \sup_{\Pi_{[0,s]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| + \sup_{\Pi_{[s,t]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| - |f(t) - f(s)| \\
&\leq \sup_{\Pi_{[0,t]}} \sum_{t_i \in \Pi} \big|f(t_{i+1}) - f(t_i)\big| - |f(t) - f(s)| \\
&= |f|(t) - |f(t) - f(s)|.
\end{aligned}$$

So we conclude that $|f|$ is non-decreasing with $|f|(t) - |f|(s) \geq |f(t) - f(s)|$. It follows from this inequality that f_1 and f_2 are non-decreasing, and $f = f_1 - f_2$ holds by construction.

We now show that this choice of f_1 and f_2 is minimal. Consider any other $\tilde{f}_1$ and $\tilde{f}_2$. Note that the bound $|f(t)| \leq \tilde{f}_1(t) + \tilde{f}_2(t)$ holds by the same argument as for the "if" statement. From the definition of f_1, f_2 and as $\tilde{f}_1(0) = \tilde{f}_2(0) = 0$,

$$\begin{aligned}
f_1(t) &= \frac{|f|(t) + f(t)}{2} \leq \frac{\tilde{f}_1(t) + \tilde{f}_2(t) + \tilde{f}_1(t) - \tilde{f}_2(t)}{2} = \tilde{f}_1(t), \\
f_2(t) &= \frac{|f|(t) - f(t)}{2} \leq \frac{\tilde{f}_1(t) + \tilde{f}_2(t) - (\tilde{f}_1(t) - \tilde{f}_2(t))}{2} = \tilde{f}_2(t),
\end{aligned}$$

as we wanted.

(b) Define $(\Delta^+ g)(t) = g(t+) - g(t) = \lim_{u \searrow t} g(u) - g(t)$ for any function $g : [0,\infty) \to \mathbb{R}$ such that the limit exists. From the equalities $f = f_1 - f_2$ and $|f| = f_1 + f_2$, and noting that f is right-continuous while f_1, f_2, f are increasing, we obtain that

$$0 = \Delta^+ f = \Delta^+ f_1 - \Delta^+ f_2, \qquad \Delta^+ |f| = \Delta^+ f_1 + \Delta^+ f_2 = 2\Delta^+ f_1.$$

We want to show that $(\Delta^+|f|)(t) = 0$ for all $t \geq 0$. Fix some $t_0 \geq 0$ and consider the functions $\tilde{f}_1(t) = f_1(t) - (\Delta^+ f_1)(t_0)\mathbb{1}_{\{t > t_0\}}$ and $\tilde{f}_2(t) = f_2(t) - (\Delta^+ f_2)(t_0)\mathbb{1}_{\{t > t_0\}}$. It is clear that $\tilde{f}_1 \leq f_1$ and $\tilde{f}_2 \leq f_2$, with $\tilde{f}_1 + \tilde{f}_2 = f$ since $\Delta^+ f_1 = \Delta^+ f_2$. Moreover, $\tilde{f}_1$ is increasing: this is clear on each interval (t_0, ∞) and $[0, t_0]$, while for $0 \leq s \leq t_0 < t$, we have that

$$f_1(t) \geq f_1(t_0+) = f_1(t_0) + (\Delta^+ f_1)(t_0) \geq f_1(s) + (\Delta^+ f_1)(t_0),$$

so $\tilde{f}_1(t) \geq \tilde{f}_1(s)$, and likewise for $\tilde{f}_2$.

However, this contradicts (a) unless $\Delta^+ f_1 \equiv \Delta^+ f_2 \equiv \Delta^+ |f| \equiv 0$, as we wanted. This shows that $|f|$ is right-continuous.

(c) Note that f_1 and f_2 are non-decreasing and $f = f_1 - f_2$. Moreover, due to (b), both f_1 and f_2 are right-continuous. Let μ_{f_1} and μ_{f_2} be the corresponding measures. Since $\mu_{f_i}([0,t]) = f_i(t) \leq |f|(t) = \mu_{|f|}([0,t])$ for every $t \geq 0$, we conclude that $\mu_{f_i} \leq \mu_{|f|}$ on $\mathcal{B}([0,\infty))$ for $i = 1,2$. Thus, we see by a monotone class argument that

$$\int_0^\infty |g(s)|\,\mu_{f_i}(ds) \leq \int_0^\infty |g(s)|\,\mu_{|f|}(ds)$$

for all measurable functions g and $i = 1,2$, and therefore $\int_0^\infty |g(s)|\,\mu_{f_1}(ds)$, $\int_0^\infty |g(s)|\,\mu_{f_2}(ds)$ are both finite whenever $\int_0^\infty |g(s)|\,\mu_{|f|}(ds)$ is.

Therefore,

$$\int_0^\infty g(s)\,\mu_{f_1}(ds) - \int_0^\infty g(s)\,\mu_{f_2}(ds)$$

is well defined.

Remark: One can show that the equality

$$\int_0^\infty g(s)\,\mu_{\tilde{f}_1}(ds) - \int_0^\infty g(s)\,\mu_{\tilde{f}_2}(ds) = \int_0^\infty g(s)df(s)$$

holds independently of the choice of $\tilde{f}_1, \tilde{f}_2$ satisfying the conditions in (a), not just the minimal f_1, f_2, as long as the integrability condition $\int_0^\infty g(s)\,\mu_{\tilde{f}_1+\tilde{f}_2}(ds) < \infty$ is satisfied. For instance, this is helpful in showing that $\int_0^\infty g(s)df(s)$ is linear in f.