Brownian Motion and Stochastic Calculus

Exercise sheet 8

Exercise 8.1 Let (N_t) be a Poisson process with rate $\lambda > 0$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

(a) Find a martingale M and a predictable finite variation process A both null at 0 such that $N_t = M_t + A_t$ for all $t \geq 0$.

(b) Compute $[M]$ and $\langle M \rangle$ for your choice in (a).

(c) Check by direct calculations that $M^2 - [M]$ is a martingale.

Solution 8.1

(a) This holds with $M_t = N_t - \lambda t$ and $A_t = \lambda t$. The equality is immediate, and A is clearly predictable and has finite variation. To show that M is a martingale, note that it is adapted and integrable (as $E[|N_t|] < \infty$) with

$$E[M_t - M_s \mid \mathcal{F}_s] = E[N_t - N_s \mid \mathcal{F}_s] - \lambda(t-s) = 0,$$

because $N_t - N_s$ is independent of $\mathcal{F}_s$ and has a Poisson distribution with parameter $\lambda(t-s)$.

(b) Since A is continuous and has finite variation, we have that $[A] \equiv [A, M] \equiv 0$. As all jumps of N are equal to 1, we obtain that

$$[M]_t = [N]_t = \sum_{0<s\leq t} (\Delta N_s)^2 = \sum_{0<s\leq t} \Delta N_s = N_t.$$

Since $\langle M \rangle$ is predictable, has finite variation and is such that $[M] - \langle M \rangle$ is a martingale, it follows by the same argument as in (a) that $\langle M \rangle_t = \lambda t$ for each $t \geq 0$.

(c) $M^2 - [M]$ is clearly adapted and integrable (as $E[N_t^2], E[|N_t|] < \infty$). Moreover,

$$E[M_t^2 - M_s^2 \mid \mathcal{F}_s] = E[(M_t - M_s)^2 \mid \mathcal{F}_s]$$

as M is a martingale. Using that $M_t - M_s = N_t - N_s - \lambda(t-s)$ and $[M] = N$, we compute

$$\begin{aligned} E[(M_t^2 - [M]_t) - (M_s^2 - [M]_s) \mid \mathcal{F}_s] &= E[(M_t - M_s)^2 \mid \mathcal{F}_s] - E[N_t - N_s \mid \mathcal{F}_s] \\ &= \mathrm{Var}[N_t - N_s] - E[N_t - N_s] \\ &= 0, \end{aligned}$$

again since $N_t - N_s$ is independent of $\mathcal{F}_s$ and has a Poisson distribution with parameter $\lambda(t-s)$.

Exercise 8.2 Let $(M_t)_{t\geq 0}$ be a local martingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, i.e. there exists a sequence of stopping times (τ_n) such that $\tau_n \nearrow \infty$ a.s. and each process $M^{\tau_n}\mathbb{1}_{\{\tau_n>0\}}$ is a martingale. Suppose that $\mathbb{F}$ satisfies the usual conditions.

Show that if $M_0 \in L^1$ and $M \geq 0$, i.e. $M_t \geq 0$ P-a.s. for all $t \geq 0$, then M is a supermartingale.

Solution 8.2 As $M_0 \in L^1$, we have that M^{τ_n} is a martingale for each n. Indeed, we have $M^{\tau_n} = M^{\tau_n}\mathbb{1}_{\{\tau_n>0\}} + N^n$, where $(N_t^n)_{t\geq 0}$ defined by $N_t^n = M_0\mathbb{1}_{\{\tau_n=0\}}$ is a martingale (N^n is clearly adapted, integrable and satisfies the martingale property).

Since $\tau_n \nearrow \infty$ a.s., $M_t^{\tau_n} = M_{\tau_n\wedge t} \to M_t$ a.s. for each $t \geq 0$. In particular, M_t is $\mathcal{F}_t$-measurable so that M is adapted. We also obtain that $M_t \in L^1$ by Fatou's lemma, since

$$E[|M_t|] = E[M_t] = E\left[\lim_{n\to\infty} M_{\tau_n\wedge t}\right] \leq \liminf_{n\to\infty} E[M_{\tau_n\wedge t}] = E[M_0] < \infty$$

by nonnegativity and the martingale property of M^{τ_n}. The supermartingale property can be similarly shown using Fatou's lemma. Indeed, for every $A \in \mathcal{F}_s$,

$$E[\mathbb{1}_A M_t] \leq \liminf_{n\to\infty} E[\mathbb{1}_A M_{\tau_n\wedge t}]$$

and therefore $E[M_t \mid \mathcal{F}_s] \leq \liminf_{n\to\infty} E[M_{\tau_n\wedge t} \mid \mathcal{F}_s]$. Since M^{τ_n} is a martingale, we have that $E[M_{\tau_n\wedge t} \mid \mathcal{F}_s] = M_{\tau_n\wedge s} \to M_s$ a.s., which shows that $E[M_t \mid \mathcal{F}_s] \leq M_s$, as we wanted.

Exercise 8.3

(a) For any $M \in \mathcal{M}^c_{0,\text{loc}}$, define as usual $M^*_t := \sup_{0\le s\le t} |M_s|$ for $t \ge 0$. Prove that for any $t \ge 0$ and $C, K > 0$, we have
$$P\big[M^*_t > C\big] \le \frac{4K}{C^2} + P\big[\langle M\rangle_t > K\big].$$
Hint: Stop $\langle M\rangle$ and use the Markov and Doob inequalities.

Remark: This result allows us to control the running supremum of M in terms of the quadratic variation of M.

(b) Let M be a right-continuous local martingale null at 0. Show that there exists a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ such that M^{τ_n} is a uniformly integrable martingale for each n.

Solution 8.3

(a) For $K > 0$, we consider the stopping time $\sigma_K := \inf\{t > 0 : \langle M\rangle_t > K\}$. Since $\langle M\rangle$ is continuous, we have that $\langle M\rangle_t \le K$ for $t \le \sigma_K$, and therefore
$$E\big[\langle M^{\sigma_K}\rangle_\infty\big] = E\big[\langle M\rangle_{\sigma_K}\big] \le K.$$
Hence, Exercise **7.3 (a)** gives that $M^{\sigma_K} \in \mathcal{H}^{2,c}_0$. We can therefore apply the Markov and Doob inequalities (noting that the constant in Doob's inequality is equal to $\big(\frac{2}{1}\big)^2 = 4$), so that
$$P\big[(M^{\sigma_K})^*_t > C\big] \le \frac{E\big[((M^{\sigma_K})^*_t)^2\big]}{C^2} \le \frac{4E\big[(M^{\sigma_K})^2_t\big]}{C^2} = \frac{4E\big[\langle M^{\sigma_K}\rangle_t\big]}{C^2} \le \frac{4K}{C^2}.$$
Now observe that
$$\{M^{\sigma_K}_t \ne M_t\} \subseteq \{\sigma_K < t\} = \{\langle M\rangle_t > K\},$$
which finally implies that
$$\begin{aligned} P\big[M^*_t > C\big] &= P\big[M^*_t > C, \sigma_K \ge t\big] + P\big[M^*_t > C, \sigma_K < t\big] \\ &\le \frac{4K}{C^2} + P\big[\langle M\rangle_t > K\big]. \end{aligned}$$

(b) Since $M \in \mathcal{M}_{0,loc}$, there exists a localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that M^{σ_n} is a martingale for each n. Consider the sequence of stopping times $\tau_n := \sigma_n \wedge n,\ n \ge 0$, so that $\tau_n \nearrow \infty$. By construction, $\tau_n \uparrow \infty$ P-a.s. and $M^{\tau_n} = (M^{\sigma_n})^n$ is a martingale for each n due to Exercise **4.2 (c)**. In particular, $M_{\tau_n} = M^{\sigma_n}_n \in \mathcal{L}^1$. Moreover, by the stopping theorem, $M_{\tau_n\wedge t} = E[M_{\tau_n}|\mathcal{F}_{\tau_n\wedge t}]$ for every $t \ge 0$, which gives uniform integrability of M^{τ_n} as in Exercise **7.3(b)**.

Exercise 8.4

(a) Let $f, g : [0,\infty) \to \mathbb{R}$ be such that g is right-continuous and has finite variation and f is g-integrable in the Lebesgue–Stieltjes sense. Show that the function $h(t) := \int_0^t f(s)\,dg(s)$ is right-continuous. Moreover, show that if g is continuous, then h is continuous.

(b) Let $f : [0,\infty) \to \mathbb{R}$ be a function with finite variation. Show that f has left and right limits, i.e. the limits $f(t-) = \lim_{s \nearrow t} f(s)$ and $f(t+) = \lim_{u \searrow t} f(u)$ exist for $t > 0$ and $t \geq 0$, respectively.

(c) Let $f, g : [0,\infty) \to \mathbb{R}$ be two right-continuous functions of finite variation. Show the integration-by-parts formula, i.e. show that for each $t > 0$,

$$f(t)\,g(t) - f(0)\,g(0) = \int_0^t f(s)\,dg(s) + \int_0^t g(s-)\,df(s) = \int_0^t f(s-)\,dg(s) + \int_0^t g(s)\,df(s).$$

(d) Show also the formula

$$f(t)\,g(t) - f(0)\,g(0) = \int_0^t f(s-)\,dg(s) + \int_0^t g(s-)\,df(s) + \sum_{0<s\leq t} \Delta f(s) \Delta g(s),$$

where $\Delta f(t) = f(t) - f(t-)$ and $\Delta g(t) = g(t) - g(t-)$.

Solution 8.4

(a) By definition of the Lebesgue–Stieltjes integral (see Exercise **7.4**) and linearity, we can without loss of generality assume that g is non-decreasing. Let μ_g be the Lebesgue–Stieltjes measure corresponding to g. Let $t \geq 0$ and $t_n \searrow t$ as $n \to \infty$. By the dominated convergence theorem,

$$\int_0^t f(s)\,dg(s) = \int_0^\infty \mathbf{1}_{[0,t]}(s)\,f(s)\,\mu_g(ds) = \lim_{n\to\infty} \int_0^\infty \mathbf{1}_{[0,t_n]}(s)\,f(s)\,\mu_g(ds) = \lim_{n\to\infty} \int_0^{t_n} f(s)\,dg(s),$$

which proves the right-continuity of $h := \int f\,dg$. Now, assume that g is continuous. Fix any $t > 0$ and let $t_n \nearrow t$. Then, by the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^{t_n} f(s)\,dg(s) &= \lim_{n\to\infty} \int_0^\infty \mathbf{1}_{[0,t_n]}(s)\,f(s)\,\mu_g(ds) \\
&= \int_0^\infty \mathbf{1}_{[0,t)}(s)\,f(s)\,\mu_g(ds) \\
&= \int_0^\infty \mathbf{1}_{[0,t]}(s)\,f(s)\,\mu_g(ds) - \int_{\{t\}} f(s)\,\mu_g(ds) \\
&= \int_0^t f(s)\,dg(s) - \int_{\{t\}} f(s)\,\mu_g(ds).
\end{aligned}$$

Thus, it remains to show that $\int_{\{t\}} f\,\mu_g(ds) = 0$. As $\mu_g(0,t] = g(t) - g(0)$, we have that $\mu_g(\{t\}) = \lim_{s\nearrow t}(g(t) - g(s)) = 0$ by the continuity of g, and hence $\int_{\{t\}} f(s)\,\mu_g(ds) = 0$. Therefore, we conclude that the continuity of g implies the continuity of $h = \int f\,dg$.

(b) From Exercise **7.4**(a), we can write $f = f_1 - f_2$, where f_1 and f_2 are non-decreasing functions. This means that the limits $f_i(t-) = \sup_{s<t} f_i(s)$ and $f_i(t+) = \inf_{u>t} f_i(u)$ are well defined for $i = 1, 2$. Therefore the limits $f(t-) = f_1(t-) - f_2(t-)$ and $f(t+) = f_1(t+) - f_2(t+)$ are also well defined.

(c) From the definition of the Lebesgue–Stieltjes integral (see Exercise **7.4**), it suffices to prove the claim for f and g non-decreasing. Let μ_f and μ_g be the corresponding Lebesgue–Stieltjes measures of f and g, respectively. Fix $t \geq 0$. From Fubini's theorem, we obtain that

$$\big(f(t) - f(0)\big)\,\big(g(t) - g(0)\big) = \int_0^t \mu_f(dr) \int_0^t \mu_g(ds) = \iint_{(0,t]\times(0,t]} \mu_f(dr)\,\mu_g(ds). \tag{1}$$

We define the domains $D_1 := \{(r,s) : 0 < r \leq s \leq t\}$ and $D_2 := \{(r,s) : 0 < s < r \leq t\}$. By definition of D_1, D_2, as $\mu_f((0,s]) = f(s) - f(0)$ and $\mu_g((0,r)) = g(r-) - g(0)$, we get

$$\begin{aligned}
&\iint_{(0,t]\times(0,t]} \mu_f(dr)\,\mu_g(ds) \\
&= \iint_{D_1} \mu_f(dr)\,\mu_g(ds) + \iint_{D_2} \mu_f(dr)\,\mu_g(ds) \\
&= \int_0^t \big(f(s) - f(0)\big)\,\mu_g(ds) + \int_0^t \big(g(r-) - g(0)\big)\,\mu_f(dr) \\
&= \int_0^t f(s)\,dg(s) - f(0)\,g(t) + f(0)\,g(0) + \int_0^t g(s-)\,df(s) - g(0)\,f(t) + g(0)\,f(0).
\end{aligned}$$

Thus, we obtain the result by plugging into (1). The second formula can be obtained by symmetry.

(d) For any $t \in [0,\infty)$, we have that $\mu_g(\{t\}) = \lim_{s\nearrow t} g((s,t]) = g(t) - g(t-) = \Delta g(t)$. In particular, $\mu_g(\{t\}) = \Delta g(t) \neq 0$ only for countably many values of t. Therefore, we can define measures $\tilde{\mu}_g = \sum_{s>0} \Delta g(s)\delta_{\{s\}}$ and $\mu_g^c = \mu_g - \tilde{\mu}_g$, and then $\mu_g^c(\{t\}) = 0$ for all $t \geq 0$. Using that $\mu_g^c(\{s : f(s) = f(s-)\}) = 0$ since the set is countable, we have that

$$\begin{aligned}
\int_0^t f(s)dg(s) &= \int_0^t f(s)d\mu_g^c(s) + \int_0^t f(s)d\tilde{\mu}_g(s) \\
&= \int_0^t f(s-)d\mu_g^c(s) + \sum_{0<s\leq t} (f(s-) + \Delta f(s))\Delta g(s) \\
&= \int_0^t f(s-)d\mu_g(s) + \sum_{0<s\leq t} \Delta f(s)\Delta g(s),
\end{aligned}$$

which shows the result.

Exercise 8.5 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let Z be a random variable which is symmetric around 0 and not in L^1, that is, $Z \stackrel{d}{=} -Z$ and $E[Z^+] = E[Z^-] = \infty$. As an example, one can let Z have a Cauchy distribution with density $f_Z(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$. Consider the discrete filtration

$$\mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_1 := \sigma(|Z|), \quad \mathcal{F}_2 := \sigma(Z)$$

and the stochastic process $(X_i)_{i=0,1,2}$ defined by $X_0 = X_1 = 0$ and $X_2 = Z$. Show that X is a local martingale with respect to $\mathbb{F}$, but not a martingale.

Solution 8.5 Since $X_2 = Z$ is not integrable, X cannot be a martingale.

To show that X is a local martingale, let $\tau_n = \mathbb{1}_{\{|Z|>n\}} + \infty \mathbb{1}_{\{|Z|\leq n\}}$ for $n \in \mathbb{N}$. We claim that each τ_n is a stopping time, $\tau_n \nearrow \infty$ a.s. and X^{τ_n} is a martingale for each n.

We check that each τ_n is a stopping time by noting that $\{\tau_n \leq 0\} = \emptyset \in \mathcal{F}_0$, while

$$\{\tau_n \leq 1\} = \{\tau_n \leq 2\} = \{|Z| > n\} \in \mathcal{F}_1,$$

and moreover it is clear that $\tau_n \nearrow \infty$ since Z is a.s. finite.

Since X is adapted, so is X^{τ_n}. For each n, we have that X^{τ_n} is integrable as it is bounded by n, noting that $X_2^{\tau_n} = Z\mathbb{1}_{\{|Z|\leq n\}}$. The martingale property is also satisfied since

$$E[X_2^{\tau_n} \mid \mathcal{F}_1] = E[Z\mathbb{1}_{\{|Z|\leq n\}} \mid \mathcal{F}_1] = \mathbb{1}_{\{|Z|\leq n\}} E[Z \mid \mathcal{F}_1] = 0,$$

using for the last equality that $E[Z \mid |Z|] = E[-Z \mid |Z|] = 0$ by symmetry of the distribution of Z.