Brownian Motion and Stochastic Calculus

Exercise sheet 9

Exercise 9.1 Let W be a Brownian motion with respect to its natural filtration. Show that

$$M_t^{(1)} = e^{t/2}\cos W_t, \quad M_t^{(2)} = tW_t - \int_0^t W_u du, \quad M_t^{(3)} = W_t^3 - 3tW_t$$

are martingales.

Remark: Prove this with a different method than the one used in Exercise **3.3**.

Hint: Recall that for each $t \geq 0$, the running maximum $W_t^* := \sup_{0\leq s\leq t}|W_s|$ has the same distribution as $|W_t|$.

Solution 9.1 We can express $M^{(1)}, M^{(2)}, M^{(3)}$ in the form

$$M_t^{(1)} = f^{(1)}(t, W_t), \quad M_t^{(2)} = f^{(2)}\left(t, W_t, \int_0^t W_u du\right), \quad M_t^{(3)} = f^{(3)}(t, W_t),$$

where

$$f^{(1)}(t,w) = e^{t/2}\cos w, \quad f^{(2)}(t,w,x) = tw - x, \quad f^{(3)}(t,w) = w^3 - 3tw$$

are C^2 functions. We note that the processes I and X defined by $I_t = t$ and $X_t = \int_0^t W_u du$, respectively, are continuous and have finite variation, while $\langle W\rangle_t = t$. Therefore, by Itô's formula,

$$\begin{aligned}
M_t^{(1)} &= M_0^{(1)} + \int_0^t \frac{\partial f^{(1)}}{\partial t}(s,W_s)ds + \int_0^t \frac{\partial f^{(1)}}{\partial w}(s,W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 f^{(1)}}{\partial w^2}(s,W_s)ds \\
&= 1 + \frac{1}{2}\int_0^t e^{s/2}\cos W_s ds - \int_0^t e^{s/2}\sin W_s dW_s - \frac{1}{2}\int_0^t e^{s/2}\cos W_s ds \\
&= 1 - \int_0^t e^{s/2}\sin W_s dW_s, \\
M_t^{(2)} &= M_0^{(2)} + \int_0^t \frac{\partial f^{(2)}}{\partial t}(s,W_s,X_s)ds + \int_0^t \frac{\partial f^{(2)}}{\partial w}(s,W_s,X_s)dW_s + \int_0^t \frac{\partial f^{(2)}}{\partial x}(s,W_s,X_s)dX_s \\
&\qquad + \frac{1}{2}\int_0^t \frac{\partial^2 f^{(2)}}{\partial w^2}(s,W_s,X_s)ds \\
&= \int_0^t W_s ds + \int_0^t s dW_s - X_t \\
&= \int_0^t s dW_s, \\
M_t^{(3)} &= M_0^{(3)} + \int_0^t \frac{\partial f^{(3)}}{\partial t}(s,W_s)ds + \int_0^t \frac{\partial f^{(3)}}{\partial w}(s,W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 f^{(3)}}{\partial w^2}(s,W_s)ds \\
&= -3\int_0^t W_s ds + \int_0^t (3W_s^2 - 3s)dW_s + 3\int_0^t W_s ds \\
&= \int_0^t (3W_s^2 - 3s)dW_s.
\end{aligned}$$

Since the integrands are continuous, hence locally bounded, we immediately get that $M^{(1)}, M^{(2)}, M^{(3)}$ are local martingales. To show that they are martingales, note that $W_t^* \overset{(d)}{=} |W_t|$. Since all moments of a Gaussian distribution are finite, the same must then be true of W_t^*. Therefore,

$$E\left[\int_0^T e^s(\sin W_s)^2 d\langle W\rangle_s\right] = E\left[\int_0^T e^s(\sin W_s)^2 ds\right] \leq Te^T < \infty,$$

$$E\left[\int_0^T s^2 d\langle W\rangle_s\right] = E\left[\int_0^T s^2 ds\right] = T^3/3 < \infty,$$

$$E\left[\int_0^T (3W_s^2 - 3s)^2 d\langle W\rangle_s\right] = E\left[\int_0^T (3W_s^2 - 3s)^2 ds\right] \leq TE[(3(W_T^*)^2 + 3T)^2] < \infty,$$

which shows that $(M^{(1)})^T, (M^{(2)})^T, (M^{(3)})^T \in \mathcal{H}^{2,c}$ for any $T > 0$. In particular, $M^{(1)}, M^{(2)}$ and $M^{(3)}$ are martingales.

Exercise 9.2 Let $W = (W_t)_{t\geq 0}$ be a 1-dimensional Brownian motion.

(a) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial. Show that the process $H = p(W)$ is in $L^2_{\text{loc}}(W)$, and therefore the stochastic integral $\int p(W)dW$ is well defined. Moreover, show that $\int p(W)dW$ is also a martingale.

(b) For what polynomials $p : \mathbb{R}^2 \to \mathbb{R}$ is the process $X_t = p(W_t, t)$ a martingale? Given a fixed $\lambda \in \mathbb{R}$, for what polynomials $p : \mathbb{R}^2 \to \mathbb{R}$ is the process $Y_t = e^{-\lambda t}p(W_t, t)$ a martingale?

(c) Let W' be another Brownian motion independent of W and ρ a predictable process satisfying $|\rho| \leq 1$. Prove that the process $B = (B_t)_{t\geq 0}$ given by

$$B_t = \int_0^t \rho_s\, dW_s + \int_0^t \sqrt{1-\rho_s^2}\, dW'_s$$

is a Brownian motion. Moreover, compute $\langle B, W\rangle$.

Remark: The pair (W, B) is sometimes called *correlated Brownian motion* with *instantaneous correlation* ρ.

(d) Define $Y_t = (\cos W_t, \sin W_t)^\top$ and $Z_t = (-\sin W_t, \cos W_t)^\top$. Show that Y is not a martingale, but $Z \bullet Y$ is a martingale.

Solution 9.2

(a) By linearity, it suffices to check the claim for monomials of the form $p(x) = x^m, m \in \mathbb{N}$. Since $H = p(W)$ is continuous and adapted, it is predictable and locally bounded, so $H \in L^2_{\text{loc}}(W)$. Alternatively, set $\tau_n := n$ for all $n \in \mathbb{N}$. Note that $E[W_s^{2m}] = E[W_1^{2m}]\, s^m = Cs^m < \infty$ for $s \geq 0$, so by Fubini's theorem,

$$E\left[\int_0^{\tau_n} H_s^2 d\langle W\rangle_s\right] = E\left[\int_0^n W_s^{2m} ds\right] = \int_0^n E\big[W_s^{2m}\big] ds < \infty.$$

Therefore, $\int p(W)dW$ is well defined and a local martingale. The same computations also give that, for all $T \geq 0$,

$$E\left[\Big\langle \int p(W)dW \Big\rangle_T\right] = E\left[\int_0^T W_s^{2m} d\langle W\rangle_s\right] = E\left[\int_0^T W_s^{2m} ds\right] < \infty.$$

This proves that $(\int p(W)dW)^T \in \mathcal{H}_0^{2,c}$ for all $T \geq 0$ by Exercise **8.3(a)**, implying that $\int p(W)dW$ is a true martingale.

(b) Note that the process $I_t := t$ has finite variation, and therefore $\langle I\rangle \equiv \langle W, I\rangle \equiv 0$. Moreover, $\langle W\rangle_t = t$ for $t \geq 0$.

Since any polynomial is C^2, by Itô's formula

$$X_t = X_0 + \int_0^t \partial_x p(W_s, s)dW_s + \int_0^t \partial_t p(W_s, s)ds + \frac{1}{2}\int_0^t \partial_{xx} p(W_s, s)ds.$$

Note that $\partial_x p$ is a polynomial, therefore by (a), $\int_0^\cdot \partial_x p(W_s, s)dW_s$ is a true martingale for any p. It follows that X is a martingale if $\partial_t p = -\frac{1}{2}\partial_{xx} p$.

Conversely, suppose that X is a martingale. Then, by Proposition 4.1.4 from the lecture notes,

$$\int_0^t \left(\partial_t p(W_s, s) + \frac{1}{2}\partial_{xx} p(W_s, s)\right) ds = 0, \quad t \geq 0,$$

since the integral is a continuous martingale with finite variation. By continuity of the integrand in t, it follows that $\partial_t p(W_t,t)+\frac{1}{2}\partial_{xx}p(W_t,t)=0$ a.s. for all $t\geq 0$. In particular, it holds that

$$E\left[\left(\partial_t p(W_t,t)+\frac{1}{2}\partial_{xx}p(W_t,t)\right)^2\right]=0,\quad \text{for all } t\geq 0.$$

Letting g_t be the Gaussian density function of W_t, we have that

$$\int_{-\infty}^{\infty}\left(\partial_t p(x,t)+\frac{1}{2}\partial_{xx}p(x,t)\right)^2 g_t(x)dx=0,\quad \text{for all } t\geq 0.$$

Since the integrand is nonnegative, $\partial_t p(x,t)+\frac{1}{2}\partial_{xx}p(x,t)=0$ for λ-almost all $x\in\mathbb{R}$ and all $t\geq 0$. By continuity, $\partial_t p(x,t)=-\frac{1}{2}\partial_{xx}p(x,t)$ must hold for all $x\in\mathbb{R}, t\geq 0$. By properties of polynomials (e.g., since polynomials are analytic), we obtain that $\partial_t p\equiv -\frac{1}{2}\partial_{xx}p$.

Consider now the function $f(x,t)=e^{-\lambda t}p(x,t)$, which is C^2. By Itô's formula,

$$Y_t=X_0+\int_0^t\partial_x f(W_s,s)dW_s+\int_0^t\left(e^{-\lambda s}\partial_t p(W_s,s)-\lambda f(W_s,s)\right)ds+\frac{1}{2}\int_0^t\partial_{xx}f(W_s,s)ds.$$

Since $|\partial_x f(W_s,s)|=|e^{-\lambda s}\partial_x p(W_s,s)|\leq(1\vee e^{-\lambda t})|\partial_x p(W_s,s)|$, we find that $\int_0^t\partial_x f(W_s,s)dW_s$ is a true martingale. By a similar argument as before, Y is a martingale if and only if

$$\begin{aligned}e^{-\lambda t}\partial_t p(x,t)&=\lambda f(x,t)-\frac{1}{2}\partial_{xx}f(x,t)\\ \Leftrightarrow \partial_t p(x,t)&=\lambda p(x,t)-\frac{1}{2}\partial_{xx}p(x,t)\quad \text{for all } x,t\in\mathbb{R}.\end{aligned}$$

(c) Since ρ is predictable and $|\rho|\leq 1$, we have that $\rho\in L^2_{\text{loc}}(W)$ and $\sqrt{1-\rho^2}\in L^2_{\text{loc}}(W')$. It follows that B is a local martingale. Moreover, for each $t\geq 0$, using bilinearity of $\langle\cdot,\cdot\rangle$ and the fact that $\langle W,W'\rangle=0$ due to independence of W and W',

$$\langle B\rangle_t=\left\langle\int\rho\,dW\right\rangle_t+\left\langle\int\sqrt{1-\rho^2}\,dW'\right\rangle_t=\int_0^t\rho_s^2\,ds+\int_0^t(1-\rho_s^2)\,ds=t,$$

and so Lévy's characterisation of Brownian motion yields that B is a Brownian motion. Finally,

$$\langle B,W\rangle_t=\int_0^t\rho_s\,d\langle W,W\rangle_s+\int_0^t\sqrt{1-\rho_s^2}\,d\langle W',W\rangle_s=\int_0^t\rho_s\,ds.$$

(d) By Itô's formula, we have that

$$\begin{aligned}\cos W_t&=1-\int_0^t\sin W_s dW_s-\frac{1}{2}\int_0^t\cos W_s ds,\\ \sin W_t&=\int_0^t\cos W_s dW_s-\frac{1}{2}\int_0^t\sin W_s ds.\end{aligned}$$

In vector form, we obtain that

$$Y_t=Y_0+\int_0^t Z_s dW_s-\frac{1}{2}\int_0^t Y_s ds.$$

Since the integrand is bounded, it is clear that $\int_0^t Z_s dW_s$ is a martingale. But the term $\frac{1}{2}\int_0^t Y_s ds$ is nonzero, continuous and has finite variation, and therefore Y is not a martingale. By associativity of stochastic integrals, we have that

$$(Z\bullet Y)_t=\int_0^t Z_s^\top Z_s dW_s-\frac{1}{2}\int_0^t Z_s^\top Y_s ds=W_t,$$

since $Z_s^\top Z_s\equiv 1$ and $Z_s^\top Y_s\equiv 0$. In particular, $Z\bullet Y$ is a martingale.

Exercise 9.3 Let $M \in \mathcal{H}_0^{2,c}$. Show that $b\mathcal{E}$ is dense in $L^2(M)$.

Hint: Let $\overline{\Omega} = \Omega \times [0,\infty)$ be equipped with the predictable σ-algebra $\mathcal{P}$. Let $C = E[M_\infty^2]$ and consider the probability measure $P_M = C^{-1} P \otimes [M]$ on $(\overline{\Omega}, \mathcal{P})$. Let $(\Pi_n)_{n\in\mathbb{N}}$ be an increasing sequence of partitions of $[0,\infty)$ with $\lim_{n\to\infty} |\Pi_n| = 0$. Use the martingale convergence theorem on $(\overline{\Omega}, \mathcal{P}, P_M)$ with respect to the discrete filtration $(\mathcal{P}_n)_{n\in\mathbb{N}}$ defined by

$$\mathcal{P}_n = \sigma(\{A_i \times (t_i, t_{i+1}] : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}).$$

Solution 9.3 We first note that $L^2(M) = L^2_{P_M}$, since both are equal to the set of (equivalence classes of) predictable processes $\tilde{H}$ such that

$$\|\tilde{H}\|^2_{L^2(M)} = E\left[\int_0^\infty \tilde{H}_s^2 d\langle M\rangle_s\right] = CE_M[\tilde{H}^2] = C\|\tilde{H}\|^2_{L^2_{P_M}} < \infty.$$

Let $H \in L^2(M)$. We want to approximate H in $L^2(M)$ by elements of $b\mathcal{E}$. Since $H\mathbb{1}_{\{|H|\le n\}} \to H$ in $L^2(M)$, we only need to approximate each $H\mathbb{1}_{\{|H|\le n\}}$. Thus, we assume w.l.o.g. that H is bounded.

Define a P_M-martingale (H_n) adapted to $(\mathcal{P}_n)$ by $H^n := E_M[H \mid \mathcal{P}_n]$. Since $H \in L^2_{P_M} = L^2(M)$, we have that (H^n) is an $L^2_{P_M}$-bounded martingale. Let $\mathcal{P}_\infty = \sigma(\bigcup_{n\in\mathbb{N}} \mathcal{P}_n)$ and $H^\infty := E[H \mid \mathcal{P}_\infty]$. By the martingale convergence theorem, we have that $H^n \to H^\infty$ P_M-a.s. and in $L^2_{P_M}$.

We claim that $\mathcal{P}_\infty = \mathcal{P}$ and each $H^n \in b\mathcal{E}, n \in \mathbb{N}$. If this holds, then we can approximate $H = H^\infty$ in $L^2_{P_M} = L^2(M)$ as the limit of (H^n), where each $H^n \in b\mathcal{E}$. Thus, the two claims imply the result.

To show that $\mathcal{P}_\infty = \mathcal{P}$, we first note that each $\mathcal{P}_n \subseteq \mathcal{P}$. Indeed, let $\tilde{H} = \mathbb{1}_{A_i}\mathbb{1}_{(t_i,t_{i+1}]}$ for some $t_i \in \Pi_n$ and $A_i \in \mathcal{F}_{t_i}$. As $\tilde{H}$ is adapted and left-continuous, it is predictable, i.e., $\mathcal{P}$-measurable. Therefore, each $\mathcal{P}_n \subseteq \mathcal{P}$. Taking the union, $\mathcal{P}_\infty \subseteq \mathcal{P}$.

Conversely, to show that $\mathcal{P} \subseteq \mathcal{P}_\infty$, we show that any left-continuous adapted process $\tilde{H}$ is $\mathcal{P}_\infty$-measurable. Define

$$\tilde{H}^n := \sum_{t_i\in\Pi_n} \mathbb{1}_{(t_i,t_{i+1}]}\tilde{H}_{t_i},$$

which is $\mathcal{P}_n$-measurable, hence also $\mathcal{P}_\infty$-measurable for each n. For each $t \in [0,\infty)$ and $n \in \mathbb{N}$, we have that $\tilde{H}^n_t(\omega) = \tilde{H}_{t(n)}(\omega)$, where $t(n) = \max\{t_i \in \Pi_n : t_i < t\}$. We have that $t(n)$ is increasing in n, since (Π_n) is an increasiing sequence. Moreover, $t(n) \nearrow t$ since $|\Pi_n| \searrow 0$. As $\tilde{H}$ is left-continuous, we conclude that $\tilde{H}^n_t(\omega) \to \tilde{H}_t(\omega)$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. Therefore, as each $\tilde{H}^n$ is $\mathcal{P}_\infty$-measurable, so is $\tilde{H}$. This shows that $\mathcal{P}_\infty = \mathcal{P}$, as we wanted.

Finally, we need to prove that $H^n \in b\mathcal{E}$ for each n. We give two proofs for this.

Proof 1: Note that $H^n = E[H \mid \mathcal{P}_n]$ is bounded, as H is. Since H^n is $\mathcal{P}_n$-measurable, the result follows if we show that every bounded $\mathcal{P}_n$-measurable process belongs to $b\mathcal{E}$. We show this using the monotone class theorem. Let

$$\mathcal{M} = \{\mathbb{1}_{A_i}\mathbb{1}_{(t_i,t_{i+1}]} : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}$$

and

$$\mathcal{H} = \left\{\tilde{H} = \sum_{t_i\in\Pi_n} Z_i\mathbb{1}_{(t_i,t_{i+1}]} : Z_i \text{ bounded and } \mathcal{F}_{t_i}\text{-measurable}\right\}.$$

It is clear that $\mathcal{M}$ is closed under products, generates $\mathcal{P}_n$ and is contained in $\mathcal{H}$. Moreover, $\mathcal{H}$ is a vector space and contains 1. To see that $\mathcal{H}$ is closed under bounded monotone convergence, let $\mathcal{H} \ni \tilde{H}^m \nearrow \tilde{H}$. Then, it must be the case that $Z_i^m = \tilde{H}^m_{t_{i+1}} \nearrow \tilde{H}_{t_{i+1}} =: Z_i$, where Z_i is bounded and $\mathcal{F}_{t_i}$-measurable. Moreover, since

$$\tilde{H}_t = \lim_{m\to\infty} \tilde{H}^m_t = \lim_{m\to\infty} \tilde{H}^m_{t(n)} = \tilde{H}_{t(n)}, \quad t \ge 0,$$

we see that

$$\tilde{H} = \sum_{t_i \in \Pi_n} Z_i \mathbb{1}_{(t_i, t_{i+1}]} \in \mathcal{H}.$$

Therefore, by the monotone class theorem, $\mathcal{H}$ contains all bounded $\mathcal{P}_n$-measurable processes. Since $\mathcal{H} \subseteq b\mathcal{E}$, we have shown that every bounded $\mathcal{P}_n$-measurable process belongs to $b\mathcal{E}$, as we wanted.

Proof 2: We claim that $H^n_t = \sum_{t_i \in \Pi_n} \mathbb{1}_{\{t \in (t_i, t_{i+1}]\}} Z_i$ for each $n \in \mathbb{N}$, where

$$Z_i = \frac{E\left[\int_{t_i}^{t_{i+1}} H_u d\langle M \rangle_u \,\middle|\, \mathcal{F}_{t_i}\right]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]}.$$

If this holds, then $H^n \in b\mathcal{E}$, as Z_i is bounded and $\mathcal{F}_{t_i}$-measurable.

To show that H^n is the conditional expectation, note that H^n is $\mathcal{P}_n$-measurable. We also have that $\{A_i \times (t_i, t_{i+1}] : t_i \in \Pi_n, A_i \in \mathcal{F}_{t_i}\}$ is a π-system generating $\mathcal{P}_n$. Therefore, it suffices to check that $E_M[\mathbb{1}_{A_i \times (t_i, t_{i+1}]} H^n] = E_M[\mathbb{1}_{A_i \times (t_i, t_{i+1}]} H]$ for any $t_i \in \Pi_n$ and $A_i \in \mathcal{F}_{t_i}$. Indeed,

$$\begin{aligned}
E_M\left[\mathbb{1}_{A_i \times (t_i, t_{i+1}]} H\right] &= C^{-1} E\left[\mathbb{1}_{A_i} \int_{t_i}^{t_{i+1}} H_u d\langle M \rangle_u\right] \\
&= C^{-1} E\left[\mathbb{1}_{A_i} E\left[\int_{t_i}^{t_{i+1}} H_u d\langle M \rangle_u \,\middle|\, \mathcal{F}_{t_i}\right]\right] \\
&= C^{-1} E\left[\int_0^\infty \mathbb{1}_{A_i} \frac{E\left[\int_{t_i}^{t_{i+1}} H_u d\langle M \rangle_u \,\middle|\, \mathcal{F}_{t_i}\right]}{E[\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i} \mid \mathcal{F}_{t_i}]} \mathbb{1}_{\{s \in (t_i, t_{i+1}]\}} d\langle M \rangle_s\right] \\
&= E_M\left[\mathbb{1}_{A_i \times (t_i, t_{i+1}]} H^n\right],
\end{aligned}$$

as we wanted.

Exercise 9.4 Let $d \geq 2$, $\Omega = C([0,\infty);\mathbb{R}^d)$ and $Y = (Y_t)_{t\geq 0}$ denote the coordinate process. For each $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the unique probability measure on $(\Omega, \mathcal{Y}^0_\infty)$ under which Y is a d-dimensional Brownian motion started at x.

Let $x \in \mathbb{R}^d \setminus \{0\}$ and a, b such that $0 < a < |x| < b$. Consider the stopping times

$$\tau_a := \inf\big\{t \geq 0 : |Y_t| \leq a\big\}, \quad \tau_b := \inf\big\{t \geq 0 : |Y_t| \geq b\big\}.$$

(a) Suppose that $d \geq 3$. Show that $(X_t)_{t\geq 0}$ defined by $X_t := |Y_{\tau_a \wedge t}|^{2-d}$ is a bounded martingale under $\mathbb{P}_x$.

(b) Suppose that $d = 2$. Show that $(X_t)_{t\geq 0}$ defined by $X_t := -\log |Y_{\tau_a\wedge\tau_b\wedge t}|$ is a bounded martingale under $\mathbb{P}_x$.

(c) Let $d \geq 2$. Show that for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\mathbb{P}_x\big[Y_t \neq 0 \text{ for all } t \geq 0\big] = 1.$$

(d) Let $d \geq 3$. Show that for any $x \in \mathbb{R}^d$, we have

$$\mathbb{P}_x\Big[\lim_{t\to\infty} |Y_t| = \infty\Big] = 1.$$

Remark: The result in (d) is known as *transience of Brownian motion in* $\mathbb{R}^d$*, for* $d \geq 3$.

Solution 9.4

(a) For any C^2 function $g : (0,\infty) \to \mathbb{R}$, we can define the radial function $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ by

$$f(x) := g(|x|).$$

By straightforward calculations, we have for every $x \neq 0$ that

$$\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r), \quad \text{with } r = |x|.$$

We consider $g(r) := r^{2-d}$, which is C^2 on $(0,\infty)$, and let $f(x) := g(|x|)$. For any $x \neq 0$,

$$\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r) = (2-d)(1-d)r^{-d} + (2-d)(d-1)r^{-d} = 0,$$

which means that $f(x) = |x|^{2-d}$ is harmonic on $\mathbb{R}^d \setminus \{0\}$. By applying Itô's formula, as $\Delta f = 0$ so that the second order term vanishes, we see that $\mathbb{P}_x$-a.s. for all $t \geq 0$

$$X_t = \big|Y_{\tau_a\wedge t}\big|^{2-d} = f(Y_{\tau_a\wedge t}) = f(x) + \int_0^{\tau_a\wedge t} \nabla f(Y_s)\, dY_s,$$

which already shows that $(X_t)_{t\geq 0}$ is a local martingale, since f is C^2 and Y is continuous, so that $\nabla f(Y_s)$ is locally bounded. Moreover, as $d \geq 3$, we have that

$$0 \leq X_t = \big|Y_{\tau_a\wedge t}\big|^{2-d} = \frac{1}{|Y_{\tau_a\wedge t}|^{d-2}} \leq \frac{1}{a^{d-2}}.$$

Thus, since X is uniformly bounded, it is a true martingale.

(b) Similarly to (a), set $g(r) = -\log r$ and $f(x) = g(|x|)$. We have that

$$\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r) = \frac{1}{r^2} - \frac{1}{r^2} = 0,$$

therefore

$$X_t = -\log \left|Y_{\tau_a\wedge\tau_b\wedge t}\right| = f(Y_{\tau_a\wedge\tau_b\wedge t}) = f(x) + \int_0^{\tau_a\wedge\tau_b\wedge t} \nabla f(Y_s)\, dY_s.$$

By the same argument as in (a), X is a local martingale. Since $a \le |W_{\tau_a\wedge\tau_b\wedge t}| \le b$, we obtain that X is bounded by $-\log b \le X_t \le -\log a$ and therefore it is a true martingale.

(c) For the cases $d \ge 3$ and $d = 2$, let g and f be as defined in (a) and (b) respectively. Consider the martingale $M_t = f(Y_{\tau_a\wedge\tau_b\wedge t})$ (which is the same as X^{τ_b} in the case $d \ge 3$). Note that $\limsup_{t\to\infty} |Y_t| = \infty$; thus $\tau_b < \infty$ and also $\tau_a \wedge \tau_b < \infty$ a.s. Therefore, we have that $\lim_{t\to\infty} M_t = f(Y_{\tau_a\wedge\tau_b})$ where $|Y_{\tau_a\wedge\tau_b}| \in \{a, b\}$. Since M is bounded, hence a uniformly integrable martingale, we obtain that

$$g(|x|) = M_0 = \mathbb{E}_x\left[\lim_{t\to\infty} M_t\right] = g(a)\mathbb{P}_x[\tau_a < \tau_b] + g(b)\mathbb{P}_x[\tau_b < \tau_a].$$

Rearranging, we get

$$\mathbb{P}_x[\tau_a < \tau_b] = \frac{g(|x|) - g(b)\mathbb{P}_x[\tau_b < \tau_a]}{g(a)}.$$

Note that in both cases $d = 2$ and $d \ge 3$, it holds that $g(a) \to \infty$ as $a \searrow 0$. Therefore, taking the limit (by dominated convergence), we obtain that

$$\mathbb{P}_x[\tau_0 < \tau_b] = 0, \quad b > |x|.$$

Taking the limit again as $b \to \infty$, we obtain that

$$\mathbb{P}_x\left[\tau_0 < \lim_{b\to\infty} \tau_b\right] = 0.$$

However, since Y is locally bounded we have that $\lim_{b\to\infty} \tau_b = \infty$. Therefore, $\tau_0 = \infty$ a.s. and

$$\mathbb{P}_x\big[X_t \ne 0 \text{ for all } t \ge 0\big] = 1$$

as we wanted.

(d) By translation invariance of Brownian motion, we can assume that $x \ne 0$ without loss of generality. For $d \ge 3$, define

$$M_t = g(|Y_t|) = |Y_t|^{2-d}.$$

This is $\mathbb{P}_x$-a.s. well defined for all $t \ge 0$, since $\tau_0 = \infty$ a.s. by (c).

As in (a), M is a local martingale. In this case, while M is not bounded, it is a nonnegative local martingale, hence a supermartingale by Exercise **8.2**. Since M is a nonnegative supermartingale, it also follows by the supermartingale convergence theorem that $M_t \to M_\infty$ a.s. for some $\mathcal{F}_\infty$-measurable random variable M_∞. Noting that

$$\limsup_{t\to\infty} |Y_t| = \infty \quad \mathbb{P}_x\text{-a.s.},$$

it follows that

$$M_\infty = \lim_{t\to\infty} M_t = \liminf_{t\to\infty} |Y_t|^{2-d} = 0,$$

and therefore

$$|Y_t| = M_t^{\frac{1}{2-d}} \to \infty \quad \mathbb{P}_x\text{-a.s.}$$

as we wanted.